

Laplace transforms

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Introduction to Laplace transforms

Why it is important to understand: Introduction to Laplace transforms

The Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach; the transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It has very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing. This chapter just gets us started in understanding some standard Laplace transforms.

At the end of this chapter, you should be able to:

- define a Laplace transform
- recognise common notations used for the Laplace transform
- derive Laplace transforms of elementary functions
- use a standard list of Laplace transforms to determine the transform of common functions

67.1 Introduction

The solution of most electrical circuit problems can be reduced ultimately to the solution of differential equations. The use of **Laplace* transforms** provides an alternative method to those discussed in Chapters 49 to 54 for solving linear differential equations.

67.2 Definition of a Laplace transform

The Laplace transform of the function $f(t)$ is defined by the integral $\int_0^{\infty} e^{-st} f(t) dt$, where s is a parameter assumed to be a real number.

Common notations used for the Laplace transform

There are various commonly used notations for the Laplace transform of $f(t)$ and these include:

(i) $\mathcal{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathcal{L}(f)$ or Lf

(iii) $\overline{f}(s)$ or $f(s)$

Also, the letter p is sometimes used instead of s as the parameter. The notation adopted in this book will be $f(t)$ for the original function and $\mathcal{L}\{f(t)\}$ for its Laplace transform.

Hence, from above:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (1)$$

67.3 Linearity property of the Laplace transform

From equation (1),

$$\begin{aligned} \mathcal{L}\{kf(t)\} &= \int_0^{\infty} e^{-st} k f(t) dt \\ &= k \int_0^{\infty} e^{-st} f(t) dt \end{aligned}$$

$$\text{i.e. } \mathcal{L}\{kf(t)\} = k\mathcal{L}\{f(t)\} \quad (2)$$

where k is any constant.

Similarly,

$$\begin{aligned} \mathcal{L}\{af(t) + bg(t)\} &= \int_0^{\infty} e^{-st} (af(t) + bg(t)) dt \\ &= a \int_0^{\infty} e^{-st} f(t) dt \\ &\quad + b \int_0^{\infty} e^{-st} g(t) dt \end{aligned}$$

$$\text{i.e. } \mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}, \quad (3)$$

where a and b are any real constants.

The Laplace transform is termed a **linear operator** because of the properties shown in equations (2) and (3).

67.4 Laplace transforms of elementary functions

Using the definition of the Laplace transform in equation (1) a number of elementary functions may be transformed. For example:



* **Who was Laplace?** Pierre-Simon, marquis de Laplace (23 March 1749–5 March 1827) was a French mathematician and astronomer who formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics. To find out more go to www.routledge.com/cw/bird

- (a)
- $f(t) = 1$
- . From equation (1),

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^{\infty} e^{-st}(1) dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= -\frac{1}{s} [e^{-s(\infty)} - e^0] = -\frac{1}{s} [0 - 1] \\ &= \frac{1}{s} \quad (\text{provided } s > 0)\end{aligned}$$

- (b)
- $f(t) = k$
- . From equation (2),

$$\begin{aligned}\mathcal{L}\{k\} &= k\mathcal{L}\{1\} \\ \text{Hence } \mathcal{L}\{k\} &= k\left(\frac{1}{s}\right) = \frac{k}{s}, \text{ from (a) above.}\end{aligned}$$

- (c)
- $f(t) = e^{at}$
- (where
- a
- is a real constant
- $\neq 0$
-).
-
- From equation (1),

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^{\infty} e^{-st}(e^{at}) dt = \int_0^{\infty} e^{-(s-a)t} dt, \\ &\quad \text{from the laws of indices,} \\ &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = \frac{1}{-(s-a)} (0 - 1) \\ &= \frac{1}{s-a} \\ &\quad (\text{provided } (s-a) > 0, \text{ i.e. } s > a)\end{aligned}$$

- (d)
- $f(t) = \cos at$
- (where
- a
- is a real constant).

From equation (1),

$$\begin{aligned}\mathcal{L}\{\cos at\} &= \int_0^{\infty} e^{-st} \cos at dt \\ &= \left[\frac{e^{-st}}{s^2 + a^2} (a \sin at - s \cos at) \right]_0^{\infty} \\ &\quad \text{by integration by parts twice (see page 485),} \\ &= \left[\frac{e^{-s(\infty)}}{s^2 + a^2} (a \sin a(\infty) - s \cos a(\infty)) \right. \\ &\quad \left. - \frac{e^0}{s^2 + a^2} (a \sin 0 - s \cos 0) \right] \\ &= \frac{s}{s^2 + a^2} \quad (\text{provided } s > 0)\end{aligned}$$

- (e)
- $f(t) = t$
- . From equation (1),

$$\mathcal{L}\{t\} = \int_0^{\infty} e^{-st} t dt = \left[\frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s} dt \right]_0^{\infty}$$

$$\begin{aligned}&= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty} \\ &\quad \text{by integration by parts} \\ &= \left[\frac{\infty e^{-s(\infty)}}{-s} - \frac{e^{-s(\infty)}}{s^2} \right] - \left[0 - \frac{e^0}{s^2} \right] \\ &= (0 - 0) - \left(0 - \frac{1}{s^2} \right)\end{aligned}$$

since $(\infty \times 0) = 0$

$$= \frac{1}{s^2} \quad (\text{provided } s > 0)$$

- (f)
- $f(t) = t^n$
- (where
- $n = 0, 1, 2, 3, \dots$
-).

By a similar method to (e) it may be shown that $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ and $\mathcal{L}\{t^3\} = \frac{(3)(2)}{s^4} = \frac{3!}{s^4}$. These results can be extended to n being any positive integer.

Thus $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ provided $s > 0$

- (g)
- $f(t) = \sinh at$
- . From Chapter 16,

$\sinh at = \frac{1}{2}(e^{at} - e^{-at})$. Hence,

$$\begin{aligned}\mathcal{L}\{\sinh at\} &= \mathcal{L}\left\{ \frac{1}{2}e^{at} - \frac{1}{2}e^{-at} \right\} \\ &= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\} \\ &\quad \text{from equations (2) and (3)}\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2} \left[\frac{1}{s-a} \right] - \frac{1}{2} \left[\frac{1}{s+a} \right] \\ &\quad \text{from (c) above} \\ &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] \\ &= \frac{a}{s^2 - a^2} \quad (\text{provided } s > a)\end{aligned}$$

A list of elementary standard Laplace transforms are summarised in Table 67.1.

67.5 Worked problems on standard Laplace transforms

Problem 1. Using a standard list of Laplace transforms, determine the following:

(a) $\mathcal{L}\left\{1 + 2t - \frac{1}{3}t^4\right\}$ (b) $\mathcal{L}\{5e^{2t} - 3e^{-t}\}$

Table 67.1 Elementary standard Laplace transforms

Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
(i) 1	$\frac{1}{s}$
(ii) k	$\frac{k}{s}$
(iii) e^{at}	$\frac{1}{s-a}$
(iv) $\sin at$	$\frac{a}{s^2 + a^2}$
(v) $\cos at$	$\frac{s}{s^2 + a^2}$
(vi) t	$\frac{1}{s^2}$
(vii) t^2	$\frac{2!}{s^3}$
(viii) t^n ($n = 1, 2, 3, \dots$)	$\frac{n!}{s^{n+1}}$
(ix) $\cosh at$	$\frac{s}{s^2 - a^2}$
(x) $\sinh at$	$\frac{a}{s^2 - a^2}$

$$\begin{aligned}
 \text{(a)} \quad \mathcal{L}\left\{1 + 2t - \frac{1}{3}t^4\right\} &= \mathcal{L}\{1\} + 2\mathcal{L}\{t\} - \frac{1}{3}\mathcal{L}\{t^4\} \\
 &\quad \text{from equations (2) and (3)} \\
 &= \frac{1}{s} + 2\left(\frac{1}{s^2}\right) - \frac{1}{3}\left(\frac{4!}{s^{4+1}}\right) \\
 &\quad \text{from (i), (vi) and (viii) of Table 67.1} \\
 &= \frac{1}{s} + \frac{2}{s^2} - \frac{1}{3}\left(\frac{4 \cdot 3 \cdot 2 \cdot 1}{s^5}\right) \\
 &= \frac{1}{s} + \frac{2}{s^2} - \frac{8}{s^5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \mathcal{L}\{5e^{2t} - 3e^{-t}\} &= 5\mathcal{L}\{e^{2t}\} - 3\mathcal{L}\{e^{-t}\} \\
 &\quad \text{from equations (2) and (3)} \\
 &= 5\left(\frac{1}{s-2}\right) - 3\left(\frac{1}{s-(-1)}\right) \\
 &\quad \text{from (iii) of Table 67.1}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{5}{s-2} - \frac{3}{s+1} \\
 &= \frac{5(s+1) - 3(s-2)}{(s-2)(s+1)} \\
 &= \frac{2s+11}{s^2-s-2}
 \end{aligned}$$

Problem 2. Find the Laplace transforms of:(a) $6 \sin 3t - 4 \cos 5t$ (b) $2 \cosh 2\theta - \sinh 3\theta$

$$\begin{aligned}
 \text{(a)} \quad \mathcal{L}\{6 \sin 3t - 4 \cos 5t\} &= 6\mathcal{L}\{\sin 3t\} - 4\mathcal{L}\{\cos 5t\} \\
 &= 6\left(\frac{3}{s^2 + 3^2}\right) - 4\left(\frac{s}{s^2 + 5^2}\right) \\
 &\quad \text{from (iv) and (v) of Table 67.1} \\
 &= \frac{18}{s^2 + 9} - \frac{4s}{s^2 + 25}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \mathcal{L}\{2 \cosh 2\theta - \sinh 3\theta\} &= 2\mathcal{L}\{\cosh 2\theta\} - \mathcal{L}\{\sinh 3\theta\} \\
 &= 2\left(\frac{s}{s^2 - 2^2}\right) - \left(\frac{3}{s^2 - 3^2}\right) \\
 &\quad \text{from (ix) and (x) of Table 67.1} \\
 &= \frac{2s}{s^2 - 4} - \frac{3}{s^2 - 9}
 \end{aligned}$$

Problem 3. Prove that

$$\text{(a)} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} \quad \text{(b)} \quad \mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$\text{(c)} \quad \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

(a) From equation (1),

$$\begin{aligned}
 \mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at \, dt \\
 &= \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty \\
 &\quad \text{by integration by parts} \\
 &= \frac{1}{s^2 + a^2} [e^{-s(\infty)} (-s \sin a(\infty) \\
 &\quad - a \cos a(\infty)) - e^0 (-s \sin 0 \\
 &\quad - a \cos 0)]
 \end{aligned}$$

$$= \frac{1}{s^2 + a^2} [(0) - 1(0 - a)]$$

$$= \frac{a}{s^2 + a^2} \text{ (provided } s > 0)$$

(b) From equation (1),

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 dt$$

$$= \left[\frac{t^2 e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^\infty$$

by integration by parts twice

$$= \left[(0 - 0 - 0) - \left(0 - 0 - \frac{2}{s^3} \right) \right]$$

$$= \frac{2}{s^3} \text{ (provided } s > 0)$$

(c) From equation (1),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{ \frac{1}{2}(e^{at} + e^{-at}) \right\}$$

from Chapter 16

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\} \text{ from}$$

equations (2) and (3)

$$= \frac{1}{2} \left(\frac{1}{s-a} \right) + \frac{1}{2} \left(\frac{1}{s-(-a)} \right)$$

from (iii) of Table 67.1

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{(s+a) + (s-a)}{(s-a)(s+a)} \right]$$

$$= \frac{s}{s^2 - a^2} \text{ (provided } s > a)$$

Problem 4. Determine the Laplace transforms of:

(a) $\sin^2 t$ (b) $\cosh^2 3x$

(a) Since $\cos 2t = 1 - 2\sin^2 t$ then

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t). \text{ Hence,}$$

$$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{ \frac{1}{2}(1 - \cos 2t) \right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2} \left(\frac{1}{s} \right) - \frac{1}{2} \left(\frac{s}{s^2 + 2^2} \right)$$

from (i) and (v) of Table 67.1

$$= \frac{(s^2 + 4) - s^2}{2s(s^2 + 4)} = \frac{4}{2s(s^2 + 4)}$$

$$= \frac{2}{s(s^2 + 4)}$$

(b) Since $\cosh 2x = 2\cosh^2 x - 1$ then

$$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x) \text{ from Chapter 16.}$$

$$\text{Hence } \cosh^2 3x = \frac{1}{2}(1 + \cosh 6x)$$

$$\text{Thus } \mathcal{L}\{\cosh^2 3x\} = \mathcal{L}\left\{ \frac{1}{2}(1 + \cosh 6x) \right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cosh 6x\}$$

$$= \frac{1}{2} \left(\frac{1}{s} \right) + \frac{1}{2} \left(\frac{s}{s^2 - 6^2} \right)$$

$$= \frac{2s^2 - 36}{2s(s^2 - 36)} = \frac{s^2 - 18}{s(s^2 - 36)}$$

Problem 5. Find the Laplace transform of $3\sin(\omega t + \alpha)$, where ω and α are constants.

Using the compound angle formula for $\sin(A+B)$, from Chapter 19, $\sin(\omega t + \alpha)$ may be expanded to $(\sin \omega t \cos \alpha + \cos \omega t \sin \alpha)$. Hence,

$$\mathcal{L}\{3\sin(\omega t + \alpha)\}$$

$$= \mathcal{L}\{3(\sin \omega t \cos \alpha + \cos \omega t \sin \alpha)\}$$

$$= 3\cos \alpha \mathcal{L}\{\sin \omega t\} + 3\sin \alpha \mathcal{L}\{\cos \omega t\},$$

since α is a constant

$$= 3\cos \alpha \left(\frac{\omega}{s^2 + \omega^2} \right) + 3\sin \alpha \left(\frac{s}{s^2 + \omega^2} \right)$$

from (iv) and (v) of Table 67.1

$$= \frac{3}{(s^2 + \omega^2)} (\omega \cos \alpha + s \sin \alpha)$$

Now try the following Practice Exercise

Practice Exercise 247 Introduction to Laplace transforms (Answers on page 866)

Determine the Laplace transforms in Problems 1 to 9.

1. (a) $2t - 3$ (b) $5t^2 + 4t - 3$

2. (a) $\frac{t^3}{24} - 3t + 2$ (b) $\frac{t^5}{15} - 2t^4 + \frac{t^2}{2}$

3. (a) $5e^{3t}$ (b) $2e^{-2t}$

4. (a) $4\sin 3t$ (b) $3\cos 2t$

5. (a) $7\cosh 2x$ (b) $\frac{1}{3}\sinh 3t$

6. (a) $2\cos^2 t$ (b) $3\sin^2 2x$

7. (a) $\cosh^2 t$ (b) $2\sinh^2 2\theta$

8. $4\sin(at + b)$, where a and b are constants.

9. $3\cos(\omega t - \alpha)$, where ω and α are constants.

10. Show that $\mathcal{L}(\cos^2 3t - \sin^2 3t) = \frac{s}{s^2 + 36}$

For fully worked solutions to each of the problems in Practice Exercise 247 in this chapter,
go to the website:

www.routledge.com/cw/bird

Chapter 68

Properties of Laplace transforms

Why it is important to understand: Properties of Laplace transforms

As stated in the preceding chapter, the Laplace transform is a widely used integral transform with many applications in engineering, where it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. The Laplace transform is also a valuable tool in solving differential equations, such as in electronic circuits, and in feedback control systems, such as in stability and control of aircraft systems. This chapter considers further transforms together with the Laplace transform of derivatives that are needed when solving differential equations.

At the end of this chapter, you should be able to:

- derive the Laplace transform of $e^{at}f(t)$
- use a standard list of Laplace transforms to determine transforms of the form $e^{at}f(t)$
- derive the Laplace transforms of derivatives
- state and use the initial and final value theorems

68.1 The Laplace transform of $e^{at}f(t)$

From Chapter 67, the definition of the Laplace transform of $f(t)$ is:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (1)$$

$$\begin{aligned} \text{Thus } \mathcal{L}\{e^{at}f(t)\} &= \int_0^{\infty} e^{-st} (e^{at} f(t)) dt \\ &= \int_0^{\infty} e^{-(s-a)t} f(t) dt \end{aligned} \quad (2)$$

(where a is a real constant)

Hence the substitution of $(s-a)$ for s in the transform shown in equation (1) corresponds to the multiplication of the original function $f(t)$ by e^{at} . This is known as a shift theorem.

68.2 Laplace transforms of the form $e^{at}f(t)$

From equation (2), Laplace transforms of the form $e^{at}f(t)$ may be deduced. For example:

(i) $\mathcal{L}\{e^{at}t^n\}$

Since $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ from (viii) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$ from equation (2) above (provided $s > a$)

(ii) $\mathcal{L}\{e^{at} \sin \omega t\}$

Since $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}$ from (iv) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at} \sin \omega t\} = \frac{\omega}{(s-a)^2 + \omega^2}$ from equation (2) (provided $s > a$)

(iii) $\mathcal{L}\{e^{at} \cosh \omega t\}$

Since $\mathcal{L}\{\cosh \omega t\} = \frac{s}{s^2 - \omega^2}$ from (ix) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at} \cosh \omega t\} = \frac{s-a}{(s-a)^2 - \omega^2}$ from equation (2) (provided $s > a$)

A summary of Laplace transforms of the form $e^{at} f(t)$ is shown in Table 68.1.

Problem 1. Determine (a) $\mathcal{L}\{2t^4 e^{3t}\}$
(b) $\mathcal{L}\{4e^{3t} \cos 5t\}$

(a) From (i) of Table 68.1,

$$\begin{aligned}\mathcal{L}\{2t^4 e^{3t}\} &= 2\mathcal{L}\{t^4 e^{3t}\} = 2 \left(\frac{4!}{(s-3)^{4+1}} \right) \\ &= \frac{2(4)(3)(2)}{(s-3)^5} = \frac{48}{(s-3)^5}\end{aligned}$$

Table 68.1 Laplace transforms of the form $e^{at} f(t)$

Function $e^{at} f(t)$ (a is a real constant)	Laplace transform $\mathcal{L}\{e^{at} f(t)\}$
(i) $e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$
(ii) $e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
(iii) $e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
(iv) $e^{at} \sinh \omega t$	$\frac{\omega}{(s-a)^2 - \omega^2}$
(v) $e^{at} \cosh \omega t$	$\frac{s-a}{(s-a)^2 - \omega^2}$

(b) From (iii) of Table 68.1,

$$\begin{aligned}\mathcal{L}\{4e^{3t} \cos 5t\} &= 4\mathcal{L}\{e^{3t} \cos 5t\} \\ &= 4 \left(\frac{s-3}{(s-3)^2 + 5^2} \right) \\ &= \frac{4(s-3)}{s^2 - 6s + 9 + 25} \\ &= \frac{4(s-3)}{s^2 - 6s + 34}\end{aligned}$$

Problem 2. Determine (a) $\mathcal{L}\{e^{-2t} \sin 3t\}$
(b) $\mathcal{L}\{3e^{\theta} \cosh 4\theta\}$

(a) From (ii) of Table 68.1,

$$\begin{aligned}\mathcal{L}\{e^{-2t} \sin 3t\} &= \frac{3}{(s-(-2))^2 + 3^2} = \frac{3}{(s+2)^2 + 9} \\ &= \frac{3}{s^2 + 4s + 4 + 9} = \frac{3}{s^2 + 4s + 13}\end{aligned}$$

(b) From (v) of Table 68.1,

$$\begin{aligned}\mathcal{L}\{3e^{\theta} \cosh 4\theta\} &= 3\mathcal{L}\{e^{\theta} \cosh 4\theta\} = \frac{3(s-1)}{(s-1)^2 - 4^2} \\ &= \frac{3(s-1)}{s^2 - 2s + 1 - 16} = \frac{3(s-1)}{s^2 - 2s - 15}\end{aligned}$$

Problem 3. Determine the Laplace transforms of
(a) $5e^{-3t} \sinh 2t$ (b) $2e^{3t}(4 \cos 2t - 5 \sin 2t)$

(a) From (iv) of Table 68.1,

$$\begin{aligned}\mathcal{L}\{5e^{-3t} \sinh 2t\} &= 5\mathcal{L}\{e^{-3t} \sinh 2t\} \\ &= 5 \left(\frac{2}{(s-(-3))^2 - 2^2} \right) \\ &= \frac{10}{(s+3)^2 - 2^2} = \frac{10}{s^2 + 6s + 9 - 4} \\ &= \frac{10}{s^2 + 6s + 5}\end{aligned}$$

(b) $\mathcal{L}\{2e^{3t}(4 \cos 2t - 5 \sin 2t)\}$

$$= 8\mathcal{L}\{e^{3t} \cos 2t\} - 10\mathcal{L}\{e^{3t} \sin 2t\}$$

$$= \frac{8(s-3)}{(s-3)^2 + 2^2} - \frac{10(2)}{(s-3)^2 + 2^2}$$

from (iii) and (ii) of Table 68.1

$$= \frac{8(s-3) - 10(2)}{(s-3)^2 + 2^2} = \frac{8s-44}{s^2-6s+13}$$

Problem 4. Show that

$$\mathcal{L}\left\{3e^{-\frac{1}{2}x} \sin^2 x\right\} = \frac{48}{(2s+1)(4s^2+4s+17)}$$

Since $\cos 2x = 1 - 2\sin^2 x$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Hence,

$$\begin{aligned} \mathcal{L}\left\{3e^{-\frac{1}{2}x} \sin^2 x\right\} &= \mathcal{L}\left\{3e^{-\frac{1}{2}x} \frac{1}{2}(1 - \cos 2x)\right\} \\ &= \frac{3}{2}\mathcal{L}\left\{e^{-\frac{1}{2}x}\right\} - \frac{3}{2}\mathcal{L}\left\{e^{-\frac{1}{2}x} \cos 2x\right\} \\ &= \frac{3}{2}\left(\frac{1}{s - \left(-\frac{1}{2}\right)}\right) - \frac{3}{2}\left(\frac{\left(s - \left(-\frac{1}{2}\right)\right)}{\left(s - \left(-\frac{1}{2}\right)\right)^2 + 2^2}\right) \end{aligned}$$

from (iii) of Table 67.1 (page 728) and (iii) of Table 68.1 above,

$$\begin{aligned} &= \frac{3}{2\left(s + \frac{1}{2}\right)} - \frac{3\left(s + \frac{1}{2}\right)}{2\left[\left(s + \frac{1}{2}\right)^2 + 2^2\right]} \\ &= \frac{3}{2s+1} - \frac{6s+3}{4\left(s^2+s+\frac{1}{4}+4\right)} \\ &= \frac{3}{2s+1} - \frac{6s+3}{4s^2+4s+17} \\ &= \frac{3(4s^2+4s+17) - (6s+3)(2s+1)}{(2s+1)(4s^2+4s+17)} \\ &= \frac{12s^2+12s+51-12s^2-6s-6s-3}{(2s+1)(4s^2+4s+17)} \\ &= \frac{48}{(2s+1)(4s^2+4s+17)} \end{aligned}$$

Now try the following Practice Exercise

Practice Exercise 248 Laplace transforms of the form $e^{at}f(t)$ (Answers on page 867)

Determine the Laplace transforms of the following functions:

- (a) $2te^{2t}$ (b) t^2e^t
- (a) $4t^3e^{-2t}$ (b) $\frac{1}{2}t^4e^{-3t}$
- (a) $e^t \cos t$ (b) $3e^{2t} \sin 2t$
- (a) $5e^{-2t} \cos 3t$ (b) $4e^{-5t} \sin t$
- (a) $2e^t \sin^2 t$ (b) $\frac{1}{2}e^{3t} \cos^2 t$
- (a) $e^t \sinh t$ (b) $3e^{2t} \cosh 4t$
- (a) $2e^{-t} \sinh 3t$ (b) $\frac{1}{4}e^{-3t} \cosh 2t$
- (a) $2e^t(\cos 3t - 3 \sin 3t)$
(b) $3e^{-2t}(\sinh 2t - 2 \cosh 2t)$

68.3 The Laplace transforms of derivatives

(a) First derivative

Let the first derivative of $f(t)$ be $f'(t)$ then, from equation (1),

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t) dt$$

From Chapter 45, when integrating by parts

$$\int u \frac{dv}{dt} dt = uv - \int v \frac{du}{dt} dt$$

When evaluating $\int_0^\infty e^{-st} f'(t) dt$,

$$\text{let } u = e^{-st} \text{ and } \frac{dv}{dt} = f'(t)$$

from which,

$$\frac{du}{dt} = -se^{-st} \text{ and } v = \int f'(t) dt = f(t)$$

$$\begin{aligned}
 \text{Hence } \int_0^{\infty} e^{-st} f'(t) dt &= [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t)(-se^{-st}) dt \\
 &= [0 - f(0)] + s \int_0^{\infty} e^{-st} f(t) dt \\
 &= -f(0) + s\mathcal{L}\{f(t)\}
 \end{aligned}$$

assuming $e^{-st} f(t) \rightarrow 0$ as $t \rightarrow \infty$, and $f(0)$ is the value of $f(t)$ at $t=0$. Hence,

$$\left. \begin{aligned} \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\ \text{or } \mathcal{L}\left\{\frac{dy}{dx}\right\} &= s\mathcal{L}\{y\} - y(0) \end{aligned} \right\} \quad (3)$$

where $y(0)$ is the value of y at $x=0$

(b) Second derivative

Let the second derivative of $f(t)$ be $f''(t)$, then from equation (1),

$$\mathcal{L}\{f''(t)\} = \int_0^{\infty} e^{-st} f''(t) dt$$

Integrating by parts gives:

$$\begin{aligned}
 \int_0^{\infty} e^{-st} f''(t) dt &= [e^{-st} f'(t)]_0^{\infty} + s \int_0^{\infty} e^{-st} f'(t) dt \\
 &= [0 - f'(0)] + s\mathcal{L}\{f'(t)\}
 \end{aligned}$$

assuming $e^{-st} f'(t) \rightarrow 0$ as $t \rightarrow \infty$, and $f'(0)$ is the value of $f'(t)$ at $t=0$. Hence

$\{f''(t)\} = -f'(0) + s[s\mathcal{L}\{f(t)\} - f(0)]$, from equation (3),

$$\left. \begin{aligned} \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \text{i.e. } \text{or } \mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) \end{aligned} \right\} \quad (4)$$

where $y'(0)$ is the value of $\frac{dy}{dx}$ at $x=0$

Equations (3) and (4) are important and are used in the solution of differential equations (see Chapter 71) and simultaneous differential equations (Chapter 72).

Problem 5. Use the Laplace transform of the first derivative to derive:

$$(a) \mathcal{L}\{k\} = \frac{k}{s} \quad (b) \mathcal{L}\{2t\} = \frac{2}{s^2}$$

$$(c) \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

From equation (3), $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

(a) Let $f(t) = k$, then $f'(t) = 0$ and $f(0) = k$

Substituting into equation (3) gives:

$$\mathcal{L}\{0\} = s\mathcal{L}\{k\} - k$$

$$\text{i.e. } k = s\mathcal{L}\{k\}$$

$$\text{Hence } \mathcal{L}\{k\} = \frac{k}{s}$$

(b) Let $f(t) = 2t$ then $f'(t) = 2$ and $f(0) = 0$

Substituting into equation (3) gives:

$$\mathcal{L}\{2\} = s\mathcal{L}\{2t\} - 0$$

$$\text{i.e. } \frac{2}{s} = s\mathcal{L}\{2t\}$$

$$\text{Hence } \mathcal{L}\{2t\} = \frac{2}{s^2}$$

(c) Let $f(t) = e^{-at}$ then $f'(t) = -ae^{-at}$ and $f(0) = 1$

Substituting into equation (3) gives:

$$\mathcal{L}\{-ae^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$-a\mathcal{L}\{e^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$1 = s\mathcal{L}\{e^{-at}\} + a\mathcal{L}\{e^{-at}\}$$

$$1 = (s+a)\mathcal{L}\{e^{-at}\}$$

$$\text{Hence } \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

Problem 6. Use the Laplace transform of the second derivative to derive

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

From equation (4),

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Let $f(t) = \cos at$, then $f'(t) = -a \sin at$ and

$f''(t) = -a^2 \cos at$, $f(0) = 1$ and $f'(0) = 0$

Substituting into equation (4) gives:

$$\mathcal{L}\{-a^2 \cos at\} = s^2 \{\cos at\} - s(1) - 0$$

$$\text{i.e. } -a^2 \mathcal{L}\{\cos at\} = s^2 \mathcal{L}\{\cos at\} - s$$

$$\text{Hence } s = (s^2 + a^2) \mathcal{L}\{\cos at\}$$

$$\text{from which, } \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

Now try the following Practice Exercise

Practice Exercise 249 Laplace transforms of derivatives (Answers on page 867)

1. Derive the Laplace transform of the first derivative from the definition of a Laplace transform. Hence derive the transform

$$\mathcal{L}\{1\} = \frac{1}{s}$$

2. Use the Laplace transform of the first derivative to derive the transforms:

$$(a) \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (b) \mathcal{L}\{3t^2\} = \frac{6}{s^3}$$

3. Derive the Laplace transform of the second derivative from the definition of a Laplace transform. Hence derive the transform

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

4. Use the Laplace transform of the second derivative to derive the transforms:

$$(a) \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$(b) \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

68.4 The initial and final value theorems

There are several Laplace transform theorems used to simplify and interpret the solution of certain problems. Two such theorems are the initial value theorem and the final value theorem.

(a) The initial value theorem states:

$$\lim_{t \rightarrow 0} [f(t)] = \lim_{s \rightarrow \infty} [s \mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{4t}$ then

$$\mathcal{L}\{3e^{4t}\} = \frac{3}{s-4}$$

from (iii) of Table 67.1, page 728.

By the initial value theorem,

$$\lim_{t \rightarrow 0} [3e^{4t}] = \lim_{s \rightarrow \infty} \left[s \left(\frac{3}{s-4} \right) \right]$$

$$\text{i.e. } 3e^0 = \infty \left(\frac{3}{\infty - 4} \right)$$

i.e. **3=3**, which illustrates the theorem.

Problem 7. Verify the initial value theorem for the voltage function $(5 + 2 \cos 3t)$ volts, and state its initial value.

Let $f(t) = 5 + 2 \cos 3t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{5 + 2 \cos 3t\} = \frac{5}{s} + \frac{2s}{s^2 + 9}$$

from (ii) and (v) of Table 67.1, page 728.

By the initial value theorem,

$$\lim_{t \rightarrow 0} [f(t)] = \lim_{s \rightarrow \infty} [s \mathcal{L}\{f(t)\}]$$

$$\begin{aligned} \text{i.e. } \lim_{t \rightarrow 0} [5 + 2 \cos 3t] &= \lim_{s \rightarrow \infty} \left[s \left(\frac{5}{s} + \frac{2s}{s^2 + 9} \right) \right] \\ &= \lim_{s \rightarrow \infty} \left[5 + \frac{2s^2}{s^2 + 9} \right] \end{aligned}$$

$$\text{i.e. } 5 + 2(1) = 5 + \frac{2\infty^2}{\infty^2 + 9} = 5 + 2$$

i.e. **7=7**, which verifies the theorem in this case.

The initial value of the voltage is thus **7V**

Problem 8. Verify the initial value theorem for the function $(2t - 3)^2$ and state its initial value.

Let $f(t) = (2t - 3)^2 = 4t^2 - 12t + 9$

Let $\mathcal{L}\{f(t)\} = \mathcal{L}(4t^2 - 12t + 9)$

$$= 4 \left(\frac{2}{s^3} \right) - \frac{12}{s^2} + \frac{9}{s}$$

from (vii), (vi) and (ii) of Table 67.1, page 728.

By the initial value theorem,

$$\begin{aligned}\lim_{t \rightarrow 0} [(2t - 3)^2] &= \lim_{s \rightarrow \infty} \left[s \left(\frac{8}{s^3} - \frac{12}{s^2} + \frac{9}{s} \right) \right] \\ &= \lim_{s \rightarrow \infty} \left[\frac{8}{s^2} - \frac{12}{s} + 9 \right]\end{aligned}$$

$$\text{i.e. } (0 - 3)^2 = \frac{8}{\infty^2} - \frac{12}{\infty} + 9$$

i.e. **9=9**, which verifies the theorem in this case.

The initial value of the given function is thus **9**

(b) The final value theorem states:

$$\lim_{t \rightarrow \infty} [f(t)] = \lim_{s \rightarrow 0} [s\mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{-4t}$ then:

$$\lim_{t \rightarrow \infty} [3e^{-4t}] = \lim_{s \rightarrow 0} \left[s \left(\frac{3}{s+4} \right) \right]$$

$$\text{i.e. } 3e^{-\infty} = (0) \left(\frac{3}{0+4} \right)$$

i.e. **0=0**, which illustrates the theorem.

Problem 9. Verify the final value theorem for the function $(2 + 3e^{-2t} \sin 4t)$ cm, which represents the displacement of a particle. State its final steady value.

$$\text{Let } f(t) = 2 + 3e^{-2t} \sin 4t$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{2 + 3e^{-2t} \sin 4t\} \\ &= \frac{2}{s} + 3 \left(\frac{4}{(s - (-2))^2 + 4^2} \right) \\ &= \frac{2}{s} + \frac{12}{(s+2)^2 + 16}\end{aligned}$$

from (ii) of Table 67.1, page 728 and (ii) of Table 68.1 on page 732.

By the final value theorem,

$$\lim_{t \rightarrow \infty} [f(t)] = \lim_{s \rightarrow 0} [s\mathcal{L}\{f(t)\}]$$

$$\begin{aligned}\text{i.e. } \lim_{t \rightarrow \infty} [2 + 3e^{-2t} \sin 4t] \\ &= \lim_{s \rightarrow 0} \left[s \left(\frac{2}{s} + \frac{12}{(s+2)^2 + 16} \right) \right] \\ &= \lim_{s \rightarrow 0} \left[2 + \frac{12s}{(s+2)^2 + 16} \right]\end{aligned}$$

$$\text{i.e. } 2 + 0 = 2 + 0$$

i.e. **2=2**, which verifies the theorem in this case.

The final value of the displacement is thus 2 cm.

The initial and final value theorems are used in pulse circuit applications where the response of the circuit for small periods of time, or the behaviour immediately after the switch is closed, are of interest. The final value theorem is particularly useful in investigating the stability of systems (such as in automatic aircraft-landing systems) and is concerned with the steady state response for large values of time t , i.e. after all transient effects have died away.

Now try the following Practice Exercise

Practice Exercise 250 Initial and final value theorems (Answers on page 867)

1. State the initial value theorem. Verify the theorem for the functions (a) $3 - 4 \sin t$ (b) $(t - 4)^2$ and state their initial values.
2. Verify the initial value theorem for the voltage functions: (a) $4 + 2 \cos t$ (b) $t - \cos 3t$ and state their initial values.
3. State the final value theorem and state a practical application where it is of use. Verify the theorem for the function $4 + e^{-2t}(\sin t + \cos t)$ representing a displacement and state its final value.
4. Verify the final value theorem for the function $3t^2 e^{-4t}$ and determine its steady state value.

For fully worked solutions to each of the problems in Practice Exercises 248 to 250 in this chapter, go to the website:

www.routledge.com/cw/bird

Inverse Laplace transforms

Why it is important to understand: Inverse Laplace transforms

Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods. This simplification in the solving of equations, coupled with the ability to directly implement electrical components in their transformed form, makes the use of Laplace transforms widespread in both electrical engineering and control systems engineering. Laplace transforms have many further applications in mathematics, physics, optics, signal processing, and probability. This chapter specifically explains how the inverse Laplace transform is determined, which can also involve the use of partial fractions. In addition, poles and zeros of transfer functions are briefly explained; these are of importance in stability and control systems.

At the end of this chapter, you should be able to:

- define the inverse Laplace transform
- use a standard list to determine the inverse Laplace transforms of simple functions
- determine inverse Laplace transforms using partial fractions
- define a pole and a zero
- determine poles and zeros for transfer functions, showing them on a pole-zero diagram

69.1 Definition of the inverse Laplace transform

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is written as $f(t) = \mathcal{L}^{-1}\{F(s)\}$

For example, since $\mathcal{L}\{1\} = \frac{1}{s}$ then $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

Similarly, since $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$ then

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2 + a^2}\right\} = \sin at, \text{ and so on.}$$

69.2 Inverse Laplace transforms of simple functions

Tables of Laplace transforms, such as the tables in Chapters 67 and 68 (see pages 728 and 732) may be used to find inverse Laplace transforms.

However, for convenience, a summary of inverse Laplace transforms is shown in Table 69.1.

Problem 1. Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 9}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{5}{3s - 1}\right\}$

Table 69.1 Inverse Laplace transforms

$F(s) = \mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
(i) $\frac{1}{s}$	1
(ii) $\frac{k}{s}$	k
(iii) $\frac{1}{s-a}$	e^{at}
(iv) $\frac{a}{s^2+a^2}$	$\sin at$
(v) $\frac{s}{s^2+a^2}$	$\cos at$
(vi) $\frac{1}{s^2}$	t
(vii) $\frac{2!}{s^3}$	t^2
(viii) $\frac{n!}{s^{n+1}}$	t^n
(ix) $\frac{a}{s^2-a^2}$	$\sinh at$
(x) $\frac{s}{s^2-a^2}$	$\cosh at$
(xi) $\frac{n!}{(s-a)^{n+1}}$	$e^{at} t^n$
(xii) $\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at} \sin \omega t$
(xiii) $\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
(xiv) $\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at} \sinh \omega t$
(xv) $\frac{s-a}{(s-a)^2-\omega^2}$	$e^{at} \cosh \omega t$

(a) From (iv) of Table 69.1,

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at,$$

$$\begin{aligned} \text{Hence } \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} \\ &= \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} \\ &= \frac{1}{3} \sin 3t \end{aligned}$$

$$\begin{aligned} \text{(b) } \mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\} &= \mathcal{L}^{-1}\left\{\frac{5}{3\left(s-\frac{1}{3}\right)}\right\} \\ &= \frac{5}{3} \mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{3}\right)}\right\} = \frac{5}{3} e^{\frac{1}{3}t} \end{aligned}$$

from (iii) of Table 69.1

Problem 2. Find the following inverse Laplace transforms:

$$\text{(a) } \mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\} \quad \text{(b) } \mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\}$$

$$\text{(a) From (vii) of Table 69.1, } \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

$$\text{Hence } \mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\} = 3 \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 3t^2$$

(b) From (viii) of Table 69.1, if s is to have a power of 4 then $n=3$

$$\text{Thus } \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3 \text{ i.e. } \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$$

$$\text{Hence } \mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{1}{2} t^3$$

Problem 3. Determine

$$\text{(a) } \mathcal{L}^{-1}\left\{\frac{7s}{s^2+4}\right\} \quad \text{(b) } \mathcal{L}^{-1}\left\{\frac{4s}{s^2-16}\right\}$$

$$\text{(a) } \mathcal{L}^{-1}\left\{\frac{7s}{s^2+4}\right\} = 7 \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = 7 \cos 2t$$

from (v) of Table 69.1

$$\text{(b) } \mathcal{L}^{-1}\left\{\frac{4s}{s^2-16}\right\} = 4 \mathcal{L}^{-1}\left\{\frac{s}{s^2-4^2}\right\}$$

$$= 4 \cosh 4t$$

from (x) of Table 69.1

Problem 4. Find

$$(a) \mathcal{L}^{-1} \left\{ \frac{3}{s^2-7} \right\} \quad (b) \mathcal{L}^{-1} \left\{ \frac{2}{(s-3)^5} \right\}$$

(a) From (ix) of Table 69.1,

$$\mathcal{L}^{-1} \left\{ \frac{a}{s^2 - a^2} \right\} = \sinh at$$

Thus

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{3}{s^2-7} \right\} &= 3\mathcal{L}^{-1} \left\{ \frac{1}{s^2 - (\sqrt{7})^2} \right\} \\ &= \frac{3}{\sqrt{7}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{7}}{s^2 - (\sqrt{7})^2} \right\} \\ &= \frac{3}{\sqrt{7}} \sinh \sqrt{7}t \end{aligned}$$

(b) From (xi) of Table 69.1,

$$\mathcal{L}^{-1} \left\{ \frac{n!}{(s-a)^{n+1}} \right\} = e^{at} t^n$$

$$\text{Thus } \mathcal{L}^{-1} \left\{ \frac{1}{(s-a)^{n+1}} \right\} = \frac{1}{n!} e^{at} t^n$$

and comparing with $\mathcal{L}^{-1} \left\{ \frac{2}{(s-3)^5} \right\}$ shows that $n=4$ and $a=3$

Hence

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{2}{(s-3)^5} \right\} &= 2\mathcal{L}^{-1} \left\{ \frac{1}{(s-3)^5} \right\} \\ &= 2 \left(\frac{1}{4!} e^{3t} t^4 \right) = \frac{1}{12} e^{3t} t^4 \end{aligned}$$

Problem 5. Determine

$$(a) \mathcal{L}^{-1} \left\{ \frac{3}{s^2-4s+13} \right\}$$

$$(b) \mathcal{L}^{-1} \left\{ \frac{2(s+1)}{s^2+2s+10} \right\}$$

$$\begin{aligned} (a) \mathcal{L}^{-1} \left\{ \frac{3}{s^2-4s+13} \right\} &= \mathcal{L}^{-1} \left\{ \frac{3}{(s-2)^2+3^2} \right\} \\ &= e^{2t} \sin 3t \end{aligned}$$

from (xii) of Table 69.1

$$\begin{aligned} (b) \mathcal{L}^{-1} \left\{ \frac{2(s+1)}{s^2+2s+10} \right\} &= \mathcal{L}^{-1} \left\{ \frac{2(s+1)}{(s+1)^2+3^2} \right\} \\ &= 2e^{-t} \cos 3t \end{aligned}$$

from (xiii) of Table 69.1

Problem 6. Determine

$$(a) \mathcal{L}^{-1} \left\{ \frac{5}{s^2+2s-3} \right\}$$

$$(b) \mathcal{L}^{-1} \left\{ \frac{4s-3}{s^2-4s-5} \right\}$$

$$\begin{aligned} (a) \mathcal{L}^{-1} \left\{ \frac{5}{s^2+2s-3} \right\} &= \mathcal{L}^{-1} \left\{ \frac{5}{(s+1)^2-2^2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{\frac{5}{2}(2)}{(s+1)^2-2^2} \right\} \\ &= \frac{5}{2} e^{-t} \sinh 2t \end{aligned}$$

from (xiv) of Table 69.1

$$\begin{aligned} (b) \mathcal{L}^{-1} \left\{ \frac{4s-3}{s^2-4s-5} \right\} &= \mathcal{L}^{-1} \left\{ \frac{4s-3}{(s-2)^2-3^2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{4(s-2)+5}{(s-2)^2-3^2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{4(s-2)}{(s-2)^2-3^2} \right\} \\ &\quad + \mathcal{L}^{-1} \left\{ \frac{5}{(s-2)^2-3^2} \right\} \end{aligned}$$

$$= 4e^{2t} \cosh 3t + \mathcal{L}^{-1} \left\{ \frac{\frac{5}{3}(3)}{(s-2)^2-3^2} \right\}$$

from (xv) of Table 69.1

$$= 4e^{2t} \cosh 3t + \frac{5}{3} e^{2t} \sinh 3t$$

from (xiv) of Table 69.1

Now try the following Practice Exercise**Practice Exercise 251 Inverse Laplace transforms of simple functions (Answers on page 867)**

Determine the inverse Laplace transforms of the following:

1. (a) $\frac{7}{s}$ (b) $\frac{2}{s-5}$
2. (a) $\frac{3}{2s+1}$ (b) $\frac{2s}{s^2+4}$
3. (a) $\frac{1}{s^2+25}$ (b) $\frac{4}{s^2+9}$
4. (a) $\frac{5s}{2s^2+18}$ (b) $\frac{6}{s^2}$
5. (a) $\frac{5}{s^3}$ (b) $\frac{8}{s^4}$
6. (a) $\frac{3s}{\frac{1}{2}s^2-8}$ (b) $\frac{7}{s^2-16}$
7. (a) $\frac{15}{3s^2-27}$ (b) $\frac{4}{(s-1)^3}$
8. (a) $\frac{1}{(s+2)^4}$ (b) $\frac{3}{(s-3)^5}$
9. (a) $\frac{s+1}{s^2+2s+10}$ (b) $\frac{3}{s^2+6s+13}$
10. (a) $\frac{2(s-3)}{s^2-6s+13}$ (b) $\frac{7}{s^2-8s+12}$
11. (a) $\frac{2s+5}{s^2+4s-5}$ (b) $\frac{3s+2}{s^2-8s+25}$

69.3 Inverse Laplace transforms using partial fractions

Sometimes the function whose inverse is required is not recognisable as a standard type, such as those listed in Table 69.1. In such cases it may be possible, by using partial fractions, to resolve the function into simpler

fractions which may be inverted on sight. For example, the function,

$$F(s) = \frac{2s-3}{s(s-3)}$$

cannot be inverted on sight from Table 69.1. However, by using partial fractions, $\frac{2s-3}{s(s-3)} \equiv \frac{1}{s} + \frac{1}{s-3}$ which may be inverted as $1 + e^{3t}$ from (i) and (iii) of Table 67.1. Partial fractions are discussed in Chapter 2, and a summary of the forms of partial fractions is given in Table 2.1 on page 16.

Problem 7. Determine $\mathcal{L}^{-1} \left\{ \frac{4s-5}{s^2-s-2} \right\}$

$$\begin{aligned} \frac{4s-5}{s^2-s-2} &\equiv \frac{4s-5}{(s-2)(s+1)} \equiv \frac{A}{s-2} + \frac{B}{s+1} \\ &\equiv \frac{A(s+1) + B(s-2)}{(s-2)(s+1)} \end{aligned}$$

Hence $4s-5 \equiv A(s+1) + B(s-2)$

When $s=2$, $3=3A$, from which, $A=1$

When $s=-1$, $-9=-3B$, from which, $B=3$

$$\begin{aligned} \text{Hence } \mathcal{L}^{-1} \left\{ \frac{4s-5}{s^2-s-2} \right\} &\equiv \mathcal{L}^{-1} \left\{ \frac{1}{s-2} + \frac{3}{s+1} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s-2} \right\} + \mathcal{L}^{-1} \left\{ \frac{3}{s+1} \right\} \\ &= e^{2t} + 3e^{-t}, \text{ from (iii) of Table 69.1} \end{aligned}$$

Problem 8. Find $\mathcal{L}^{-1} \left\{ \frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3} \right\}$

$$\begin{aligned} \frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3} &\equiv \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3} \\ &\equiv \frac{\left(\begin{array}{l} A(s+1)^3 + B(s-3)(s+1)^2 \\ + C(s-3)(s+1) + D(s-3) \end{array} \right)}{(s-3)(s+1)^3} \end{aligned}$$

Hence

$$\begin{aligned} 3s^3+s^2+12s+2 &\equiv A(s+1)^3 + B(s-3)(s+1)^2 \\ &\quad + C(s-3)(s+1) + D(s-3) \end{aligned}$$

When $s=3$, $128=64A$, from which, $A=2$

When $s=-1$, $-12=-4D$, from which, $D=3$

Equating s^3 terms gives: $3=A+B$ from which, $B=1$

Equating constant terms gives:

$$2 = A - 3B - 3C - 3D$$

$$\text{i.e. } 2 = 2 - 3 - 3C - 9$$

from which, $3C = -12$ and $C = -4$

Hence

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{3s^3 + s^2 + 12s + 2}{(s-3)(s+1)^3} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{2}{s-3} + \frac{1}{s+1} - \frac{4}{(s+1)^2} + \frac{3}{(s+1)^3} \right\} \\ = 2e^{3t} + e^{-t} - 4e^{-t}t + \frac{3}{2}e^{-t}t^2 \\ \text{from (iii) and (xi) of Table 69.1} \end{aligned}$$

Problem 9. Determine

$$\mathcal{L}^{-1} \left\{ \frac{5s^2 + 8s - 1}{(s+3)(s^2+1)} \right\}$$

$$\begin{aligned} \frac{5s^2 + 8s - 1}{(s+3)(s^2+1)} &\equiv \frac{A}{s+3} + \frac{Bs+C}{s^2+1} \\ &\equiv \frac{A(s^2+1) + (Bs+C)(s+3)}{(s+3)(s^2+1)} \end{aligned}$$

$$\text{Hence } 5s^2 + 8s - 1 \equiv A(s^2+1) + (Bs+C)(s+3)$$

When $s=-3$, $20=10A$, from which, $A=2$

Equating s^2 terms gives: $5=A+B$, from which, $B=3$, since $A=2$

Equating s terms gives: $8=3B+C$, from which, $C=-1$, since $B=3$

$$\begin{aligned} \text{Hence } \mathcal{L}^{-1} \left\{ \frac{5s^2 + 8s - 1}{(s+3)(s^2+1)} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{2}{s+3} + \frac{3s-1}{s^2+1} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{2}{s+3} \right\} + \mathcal{L}^{-1} \left\{ \frac{3s}{s^2+1} \right\} \\ - \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} \end{aligned}$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

from (iii), (v) and (iv) of Table 69.1

Problem 10. Find $\mathcal{L}^{-1} \left\{ \frac{7s+13}{s(s^2+4s+13)} \right\}$

$$\begin{aligned} \frac{7s+13}{s(s^2+4s+13)} &\equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13} \\ &\equiv \frac{A(s^2+4s+13) + (Bs+C)(s)}{s(s^2+4s+13)} \end{aligned}$$

$$\text{Hence } 7s+13 \equiv A(s^2+4s+13) + (Bs+C)(s).$$

When $s=0$, $13=13A$, from which, $A=1$

Equating s^2 terms gives: $0=A+B$, from which, $B=-1$

Equating s terms gives: $7=4A+C$, from which, $C=3$

$$\begin{aligned} \text{Hence } \mathcal{L}^{-1} \left\{ \frac{7s+13}{s(s^2+4s+13)} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{-s+3}{s^2+4s+13} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{-s+3}{(s+2)^2+3^2} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{-(s+2)+5}{(s+2)^2+3^2} \right\} \\ \equiv \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+3^2} \right\} \\ + \mathcal{L}^{-1} \left\{ \frac{5}{(s+2)^2+3^2} \right\} \end{aligned}$$

$$\equiv 1 - e^{-2t} \cos 3t + \frac{5}{3} e^{-2t} \sin 3t$$

from (i), (xiii) and (xii) of Table 69.1

Now try the following Practice Exercise

Practice Exercise 252 Inverse Laplace transforms using partial fractions (Answers on page 867)

Use partial fractions to find the inverse Laplace transforms of the following functions:

$$1. \frac{11-3s}{s^2+2s-3}$$

$$2. \frac{2s^2 - 9s - 35}{(s+1)(s-2)(s+3)}$$

$$3. \frac{5s^2 - 2s - 19}{(s+3)(s-1)^2}$$

$$4. \frac{3s^2 + 16s + 15}{(s+3)^3}$$

$$5. \frac{7s^2 + 5s + 13}{(s^2 + 2)(s+1)}$$

$$6. \frac{3 + 6s + 4s^2 - 2s^3}{s^2(s^2 + 3)}$$

$$7. \frac{26 - s^2}{s(s^2 + 4s + 13)}$$

69.4 Poles and zeros

It was seen in the previous section that Laplace transforms, in general, have the form $f(s) = \frac{\phi(s)}{\theta(s)}$. This is the same form as most transfer functions for engineering systems, a **transfer function** being one that relates the response at a given pair of terminals to a source or stimulus at another pair of terminals.

Let a function in the s domain be given by: $f(s) = \frac{\phi(s)}{(s-a)(s-b)(s-c)}$ where $\phi(s)$ is of less degree than the denominator.

Poles: The values $a, b, c, \dots$ that makes the denominator zero, and hence $f(s)$ infinite, are called the system poles of $f(s)$.

If there are no repeated factors, the poles are **simple poles**.

If there are repeated factors, the poles are **multiple poles**.

Zeros: Values of s that make the numerator $\phi(s)$ zero, and hence $f(s)$ zero, are called the system zeros of $f(s)$.

For example: $\frac{s-4}{(s+1)(s-2)}$ has simple poles at $s=-1$ and $s=+2$, and a zero at $s=4$. $\frac{s+3}{(s+1)^2(2s+5)}$ has a simple pole at $s=-\frac{5}{2}$ and double poles at $s=-1$, and

a zero at $s=-3$ and $\frac{s+2}{s(s-1)(s+4)(2s+1)}$ has simple poles at $s=0, +1, -4$, and $-\frac{1}{2}$ and a zero at $s=-2$.

Pole-zero diagram

The poles and zeros of a function are values of complex frequency s and can therefore be plotted on the complex frequency or s -plane. The resulting plot is the **pole-zero diagram** or **pole-zero map**. On the rectangular axes, the real part is labelled the σ -axis and the imaginary part the $j\omega$ -axis.

The location of a pole in the s -plane is denoted by a cross ($\times$) and the location of a zero by a small circle ($\circ$). This is demonstrated in the following examples.

From the pole-zero diagram it may be determined that the magnitude of the transfer function will be larger when it is closer to the poles and smaller when it is close to the zeros. This is important in understanding what the system does at various frequencies and is crucial in the study of **stability** and **control theory** in general.

Problem 11. Determine for the transfer function:

$$R(s) = \frac{400(s+10)}{s(s+25)(s^2+10s+125)}$$

(a) the zero and (b) the poles. Show the poles and zero on a pole-zero diagram.

(a) For the numerator to be zero, $(s+10)=0$

Hence, **$s=-10$ is a zero** of $R(s)$

(b) For the denominator to be zero, $s=0$ or $s=-25$ or $s^2+10s+125=0$

Using the quadratic formula.

$$\begin{aligned} s &= \frac{-10 \pm \sqrt{10^2 - 4(1)(125)}}{2} = \frac{-10 \pm \sqrt{-400}}{2} \\ &= \frac{-10 \pm j20}{2} \\ &= (-5 \pm j10) \end{aligned}$$

Hence, **poles occur at $s=0$, $s=-25$, $(-5+j10)$ and $(-5-j10)$**

The pole-zero diagram is shown in Figure 69.1.

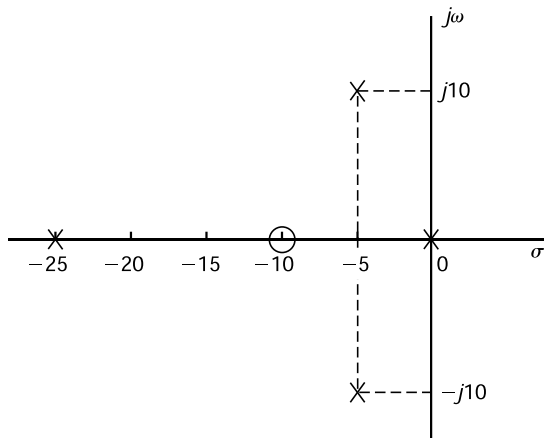


Figure 69.1

Problem 12. Determine the poles and zeros for the function: $F(s) = \frac{(s+3)(s-2)}{(s+4)(s^2+2s+2)}$ and plot them on a pole-zero map.

For the numerator to be zero, $(s+3)=0$ and $(s-2)=0$, hence **zeros occur at $s=-3$ and at $s=+2$** Poles occur when the denominator is zero, i.e. when $(s+4)=0$, i.e. **$s=-4$** , and when $s^2+2s+2=0$

$$\begin{aligned} \text{i.e. } s &= \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm j2}{2} = (-1+j) \text{ or } (-1-j) \end{aligned}$$

The poles and zeros are shown on the pole-zero map of $F(s)$ in Figure 69.2.

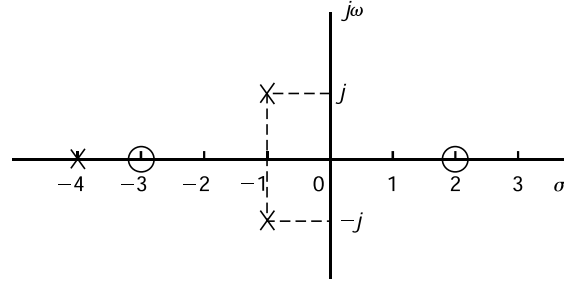


Figure 69.2

It is seen from these problems that poles and zeros are always real or complex conjugate.

Now try the following Practice Exercise

Practice Exercise 253 Poles and zeros
(Answers on page 867)

1. Determine for the transfer function:

$$R(s) = \frac{50(s+4)}{s(s+2)(s^2-8s+25)}$$

(a) the zero and (b) the poles. Show the poles and zeros on a pole-zero diagram.

2. Determine the poles and zeros for the function:

$$F(s) = \frac{(s-1)(s+2)}{(s+3)(s^2-2s+5)}$$
 and plot them on a pole-zero map.

3. For the function $G(s) = \frac{s-1}{(s+2)(s^2+2s+5)}$ determine the poles and zeros and show them on a pole-zero diagram.

4. Find the poles and zeros for the transfer function: $H(s) = \frac{s^2-5s-6}{s(s^2+4)}$ and plot the results in the s-plane.

For fully worked solutions to each of the problems in Practice Exercises 251 to 253 in this chapter, go to the website:

www.routledge.com/cw/bird



The Laplace transform of the Heaviside function

Why it is important to understand: The Laplace transform of the Heaviside function

The Heaviside unit step function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely. It is also used in structural mechanics to describe different types of structural loads. The Heaviside function has applications in engineering where periodic functions are represented. In many physical situations things change suddenly; brakes are applied, a switch is thrown, collisions occur. The Heaviside unit function is very useful for representing sudden change.

At the end of this chapter, you should be able to:

- define the Heaviside unit step function
- use a standard list to determine the Laplace transform of $H(t - c)$
- use a standard list to determine the Laplace transform of $H(t - c) \cdot f(t - c)$
- determine the inverse transforms of Heaviside functions

70.1 Heaviside unit step function

In engineering applications, functions are frequently encountered whose values change abruptly at specified values of time t . One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time t .

The switching process can be described mathematically by the function called the **Unit Step Function** – otherwise known as the **Heaviside unit step function**. Fig. 70.1 shows a function that maintains a zero value for all values of t up to $t = c$ and a value of 1 for all values of $t \geq c$. This is the Heaviside unit step function and is denoted by:

$$f(t) = H(t - c) \quad \text{or} \quad u(t - c)$$

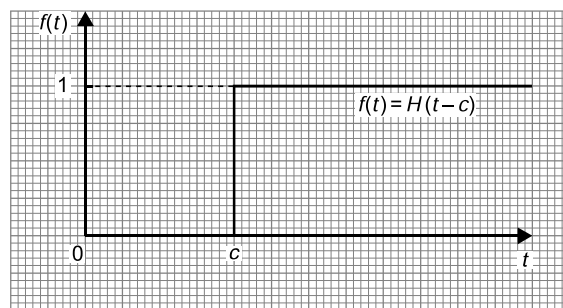


Figure 70.1

where the c indicates the value of t at which the function changes from a value of zero to a value of unity (i.e. 1).

It follows that $f(t) = H(t-5)$ is as shown in Fig. 70.2 and $f(t) = 3H(t-4)$ is as shown in Fig. 70.3.

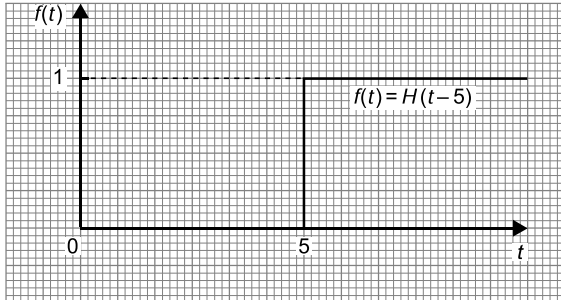


Figure 70.2

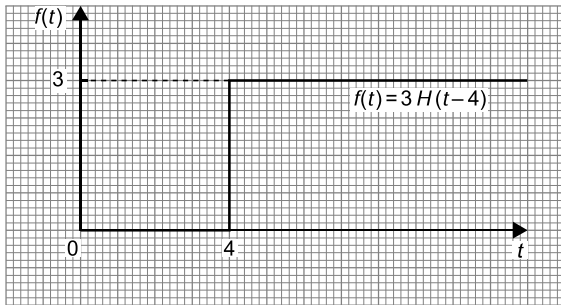


Figure 70.3

If the unit step occurs at the origin, then $c = 0$ and $f(t) = H(t-0)$, i.e. $H(t)$ as shown in Fig. 70.4.

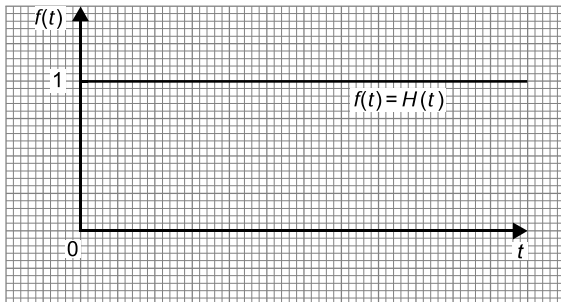


Figure 70.4

Fig. 70.5(a) shows a graph of $f(t) = t^2$; the graph shown in Fig. 70.5(b) is $f(t) = H(t-2) \cdot t^2$ where for $t < 2$, $H(t-2)t^2 = 0$ and when $t \geq 2$, $H(t-2) \cdot t^2 = t^2$. The function $H(t-2)$ suppresses the function t^2 for all values of t up to $t = 2$ and then 'switches on' the function t^2 at $t = 2$.

A common situation in an electrical circuit is for a voltage V to be applied at a particular time, say, $t = a$,

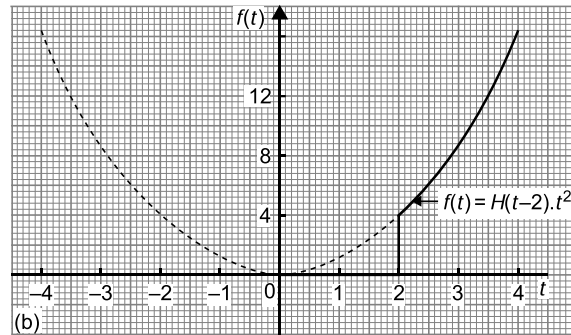
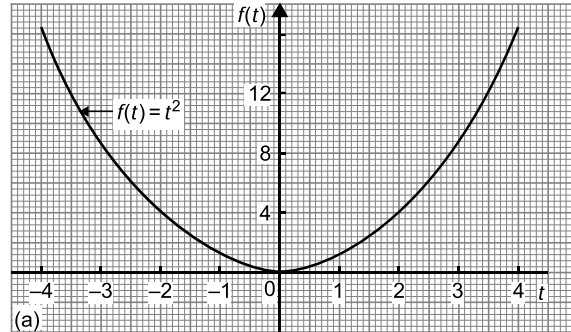


Figure 70.5

and removed later, say at $t = b$. Such a situation is written using step functions as:

$$V(t) = H(t-a) - H(t-b)$$

For example, Fig. 70.6 shows the function $f(t) = H(t-2) - H(t-5)$

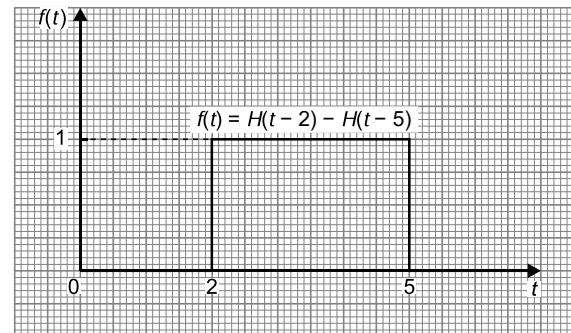


Figure 70.6

Representing the Heaviside unit step function is further explored in the following worked problems.

Problem 1. A 12 V source is switched on at time $t = 3$ s. Sketch the waveform and write the function in terms of the Heaviside step function.

The function is shown sketched in Fig. 70.7.

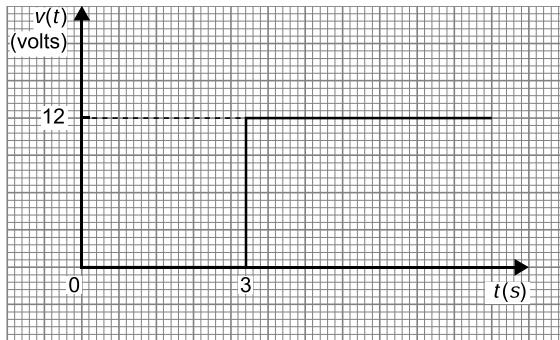


Figure 70.7

The Heaviside step function is:

$$V(t) = 12H(t - 3)$$

Problem 2. Write the function

$$V(t) = \begin{cases} 1 & \text{for } 0 < t < a \\ 0 & \text{for } t > a \end{cases}$$

in terms of the Heaviside step function and sketch the waveform.

The voltage has a value of 1 up until time $t = a$; then it is turned off.

The function is shown sketched in Fig. 70.8.

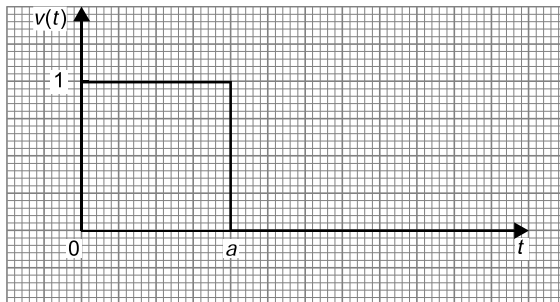


Figure 70.8

The Heaviside step function is:

$$V(t) = H(t) - H(t - a)$$

Problem 3. Sketch the graph of $f(t) = 5H(t - 2)$

A function $H(t - 2)$ has a maximum value of 1 and starts when $t = 2$.

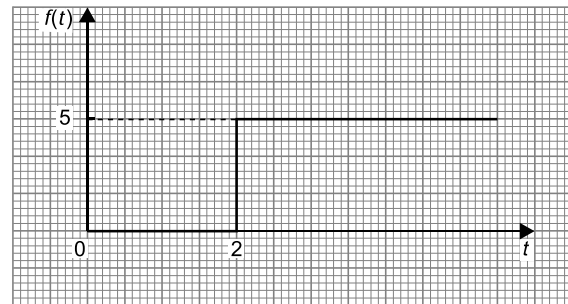


Figure 70.9

A function $5H(t - 2)$ has a maximum value of 5 and starts when $t = 2$, as shown in Fig. 70.9.

Problem 4. Sketch the graph of $f(t) = H(t - \pi/3) \cdot \sin t$

Fig. 70.10(a) shows a graph of $f(t) = \sin t$; the graph shown in Fig. 70.10(b) is $f(t) = H(t - \pi/3) \cdot \sin t$ where the graph of $\sin t$ does not 'switch on' until $t = \pi/3$

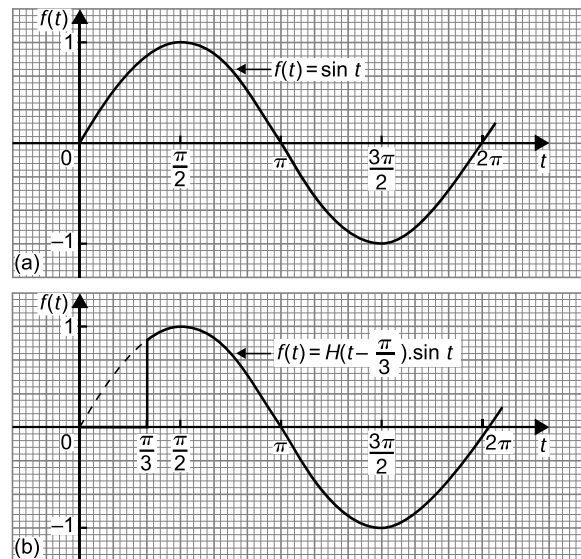


Figure 70.10

Problem 5. Sketch the graph of $f(t) = 2H(t - 2\pi/3) \cdot \sin(t - \pi/6)$

Fig. 70.11(a) shows a graph of $f(t) = 2\sin(t - \pi/6)$; the graph shown in Fig. 70.11(b) is

$$f(t) = 2H(t - 2\pi/3) \cdot \sin(t - \pi/6)$$

where the graph of $2\sin(t - \pi/6)$ does not 'switch on' until $t = 2\pi/3$

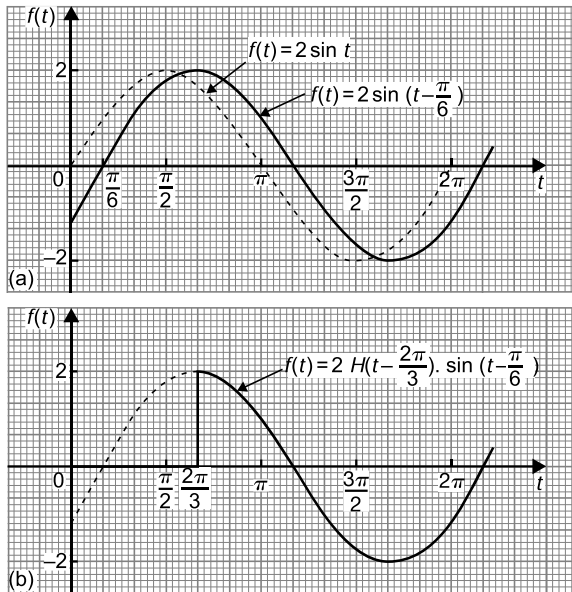


Figure 70.11

- Problem 6.** Sketch the graphs of
- $f(t) = H(t-1) \cdot e^{-t}$
 - $f(t) = [H(t-1) - H(t-3)] \cdot e^{-t}$

Fig. 70.12(a) shows a graph of $f(t) = e^{-t}$

- The graph shown in Fig. 70.12(b) is $f(t) = H(t-1) \cdot e^{-t}$ where the graph of e^{-t} does not 'switch on' until $t = 1$.
- Fig. 70.12(c) shows the graph of $f(t) = [H(t-1) - H(t-3)] \cdot e^{-t}$ where the graph of e^{-t} does not 'switch on' until $t = 1$, but then 'switches off' at $t = 3$.

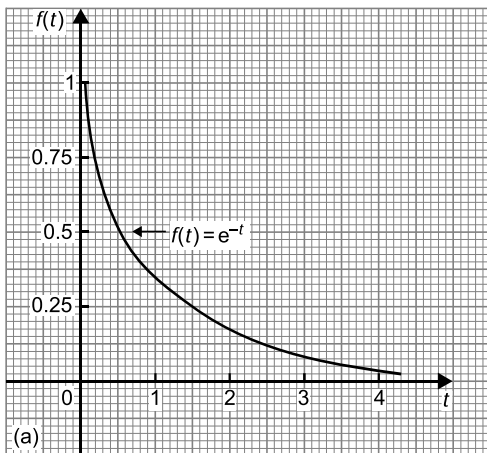


Figure 70.12

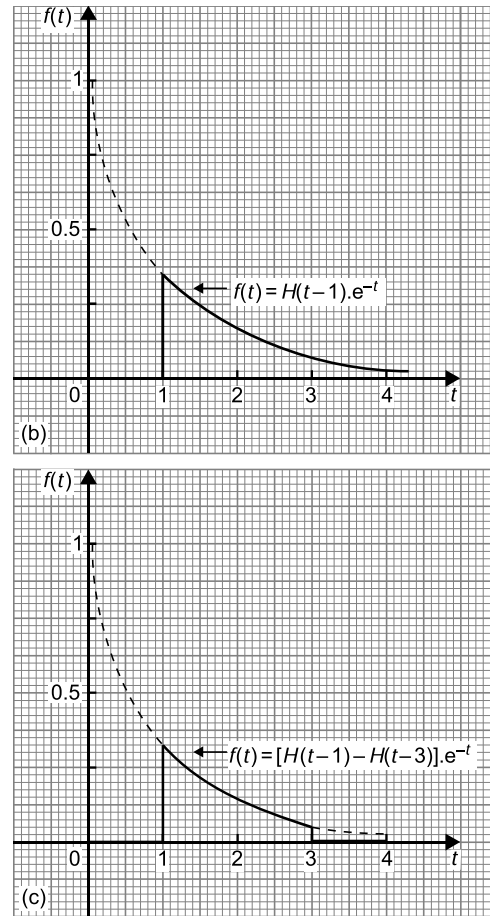


Figure 70.12 (Continued)

Now try the following Practice Exercise

Practise Exercise 254 Heaviside unit step function (Answers on page 867)

- A 6 V source is switched on at time $t = 4$ s. Write the function in terms of the Heaviside step function and sketch the waveform.
- Write the function $V(t) = \begin{cases} 2 & \text{for } 0 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$ in terms of the Heaviside step function and sketch the waveform.

In problems 3 to 12, sketch graphs of the given functions.

- $f(t) = H(t-2)$
- $f(t) = H(t)$
- $f(t) = 4H(t-1)$

6. $f(t) = 7H(t-5)$
7. $f(t) = H\left(t - \frac{\pi}{4}\right) \cdot \cos t$
8. $f(t) = 3H\left(t - \frac{\pi}{2}\right) \cdot \cos\left(t - \frac{\pi}{6}\right)$
9. $f(t) = H(t-1) \cdot t^2$
10. $f(t) = H(t-2) \cdot e^{-\frac{t}{2}}$
11. $f(t) = [H(t-2) - H(t-5)] \cdot e^{-\frac{t}{4}}$
12. $f(t) = 5H\left(t - \frac{\pi}{3}\right) \cdot \sin\left(t + \frac{\pi}{4}\right)$

70.2 Laplace transform of $H(t-c)$

From the definition of a Laplace transform,

$$\mathcal{L}\{H(t-c)\} = \int_0^{\infty} e^{-st} H(t-c) dt$$

$$\text{However, } e^{-st} H(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

$$\begin{aligned} \text{Hence, } \mathcal{L}\{H(t-c)\} &= \int_0^{\infty} e^{-st} H(t-c) dt \\ &= \int_c^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_c^{\infty} = \left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-sc}}{-s} \right] \\ &= \left[0 - \frac{e^{-sc}}{-s} \right] = \frac{e^{-sc}}{s} \end{aligned}$$

When $c=0$ (i.e. a unit step at the origin),

$$\mathcal{L}\{H(t)\} = \frac{e^{-s(0)}}{s} = \frac{1}{s}$$

$$\text{Summarising, } \mathcal{L}\{H(t)\} = \frac{1}{s} \text{ and } \mathcal{L}\{H(t-c)\} = \frac{e^{-cs}}{s}$$

$$\begin{aligned} \text{From the definition of } H(t): \mathcal{L}\{1\} &= \{1 \cdot H(t)\} \\ \mathcal{L}\{t\} &= \{t \cdot H(t)\} \end{aligned}$$

and

$$\mathcal{L}\{f(t)\} = \{f(t) \cdot H(t)\}$$

70.3 Laplace transform of $H(t-c) \cdot f(t-c)$

It may be shown that:

$$\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

This is demonstrated in the following worked problems.

Problem 7. Determine $\mathcal{L}\{4H(t-5)\}$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{4\}$ and $c=5$

Hence, $\mathcal{L}\{4H(t-5)\} = e^{-5s} \left(\frac{4}{s} \right)$ from (ii) of Table 67.1, page 728

$$= \frac{4e^{-5s}}{s}$$

Problem 8. Determine $\mathcal{L}\{H(t-3) \cdot (t-3)^2\}$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{t^2\}$ and $c=3$

Note that $F(s)$ is the transform of t^2 and not of $(t-3)^2$

Hence, $\mathcal{L}\{H(t-3) \cdot (t-3)^2\} = e^{-3s} \left(\frac{2!}{s^3} \right)$ from (vii) of Table 67.1, page 728

$$= \frac{2e^{-3s}}{s^3}$$

Problem 9. Determine $\mathcal{L}\{H(t-2) \cdot \sin(t-2)\}$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin t\}$ and $c=2$

Hence, $\mathcal{L}\{H(t-2) \cdot \sin(t-2)\} = e^{-2s} \left(\frac{1}{s^2 + 1^2} \right)$ from (iv) of Table 67.1, page 728

$$= \frac{e^{-2s}}{s^2 + 1}$$

Problem 10. Determine $\mathcal{L}\{H(t-1) \cdot \sin 4(t-1)\}$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin 4t\}$ and $c=1$

Hence, $\mathcal{L}\{H(t-1) \cdot \sin 4(t-1)\} = e^{-s} \left(\frac{4}{s^2 + 4^2} \right)$ from (iv) of Table 67.1, page 728

$$= \frac{4e^{-s}}{s^2 + 16}$$

Problem 11. Determine $\mathcal{L}\{H(t-3) \cdot e^{t-3}\}$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{e^t\}$ and $c=3$

Hence, $\mathcal{L}\{H(t-3) \cdot e^{t-3}\} = e^{-3s} \left(\frac{1}{s-1} \right)$ from (iii) of Table 67.1, page 728

$$= \frac{e^{-3s}}{s-1}$$

Problem 12. Determine

$$\mathcal{L}\left\{H\left(t-\frac{\pi}{2}\right) \cdot \cos 3\left(t-\frac{\pi}{2}\right)\right\}$$

From above, $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where in this case, $F(s) = \mathcal{L}\{\cos 3t\}$ and $c = \frac{\pi}{2}$

Hence, $\mathcal{L}\left\{\left(t-\frac{\pi}{2}\right) \cdot \cos 3\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s} \left(\frac{s}{s^2+3^2} \right)$ from (v) of Table 67.1, page 728

$$= \frac{se^{-\frac{\pi}{2}s}}{s^2+9}$$

Now try the following Practice Exercise

Practise Exercise 255 Laplace transform of $H(t-c) \cdot f(t-c)$ (Answers on page 869)

1. Determine $\mathcal{L}\{H(t-1)\}$
2. Determine $\mathcal{L}\{7H(t-3)\}$
3. Determine $\mathcal{L}\{H(t-2) \cdot (t-2)^2\}$
4. Determine $\mathcal{L}\{H(t-3) \cdot \sin(t-3)\}$
5. Determine $\mathcal{L}\{H(t-4) \cdot e^{t-4}\}$
6. Determine $\mathcal{L}\{H(t-5) \cdot \sin 3(t-5)\}$
7. Determine $\mathcal{L}\{H(t-1) \cdot (t-1)^3\}$
8. Determine $\mathcal{L}\{H(t-6) \cdot \cos 3(t-6)\}$
9. Determine $\mathcal{L}\{5H(t-5) \cdot \sinh 2(t-5)\}$
10. Determine $\mathcal{L}\left\{H\left(t-\frac{\pi}{3}\right) \cdot \cos 2\left(t-\frac{\pi}{3}\right)\right\}$
11. Determine $\mathcal{L}\{2H(t-3) \cdot e^{t-3}\}$
12. Determine $\mathcal{L}\{3H(t-2) \cdot \cosh(t-2)\}$

70.4 Inverse Laplace transforms of Heaviside functions

In the previous section it was stated that: $\mathcal{L}\{H(t-c) \cdot f(t-c)\} = e^{-cs} F(s)$ where $F(s) = \mathcal{L}\{f(t)\}$

Written in reverse, this becomes:

if $F(s) = \mathcal{L}\{f(t)\}$, then $e^{-cs} F(s) = \mathcal{L}\{H(t-c) \cdot f(t-c)\}$

This is known as the **second shift theorem** and is used when finding **inverse Laplace transforms**, as demonstrated in the following worked problems.

Problem 13. Determine $\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t-2)$

Then $\frac{3}{s} = F(s) = \mathcal{L}\{3\}$ from (ii) of Table 67.1, page 728

Hence, $\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s}\right\} = 3H(t-2)$

Problem 14. Determine the inverse of $\frac{e^{-3s}}{s^2}$

The numerator corresponds to e^{-cs} where $c = 3$. This indicates $H(t-3)$

$\frac{1}{s^2} = F(s) = \mathcal{L}\{t\}$ from (vi) of Table 67.1, page 728

Then $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = H(t-3) \cdot (t-3)$

Problem 15. Determine $\mathcal{L}^{-1}\left\{\frac{8e^{-4s}}{s^2+4}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 4$. This indicates $H(t-4)$

$\frac{8}{s^2+4}$ may be written as: $4\left(\frac{2}{s^2+2^2}\right)$

Then $4\left(\frac{2}{s^2+2^2}\right) = F(s) = \mathcal{L}\{4 \sin 2t\}$ from (iv) of Table 67.1, page 728

Hence, $\mathcal{L}^{-1}\left\{\frac{8e^{-4s}}{s^2+4}\right\} = H(t-4) \cdot 4 \sin 2(t-4)$
 $= 4H(t-4) \cdot \sin 2(t-4)$

Problem 16. Determine $\mathcal{L}^{-1} \left\{ \frac{5se^{-2s}}{s^2 + 9} \right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t - 2)$

$\frac{5s}{s^2 + 9}$ may be written as: $5 \left(\frac{s}{s^2 + 3^2} \right)$

Then $5 \left(\frac{s}{s^2 + 3^2} \right) = F(s) = \mathcal{L}\{5 \cos 3t\}$ from (v) of Table 67.1, page 728

$$\begin{aligned} \text{Hence, } \mathcal{L}^{-1} \left\{ \frac{5se^{-2s}}{s^2 + 3^2} \right\} &= H(t - 2) \cdot 5 \cos 3(t - 2) \\ &= 5H(t - 2) \cdot \cos 3(t - 2) \end{aligned}$$

Problem 17. Determine $\mathcal{L}^{-1} \left\{ \frac{7e^{-3s}}{s^2 - 1} \right\}$

Part of the numerator corresponds to e^{-cs} where $c = 3$. This indicates $H(t - 3)$

$\frac{7}{s^2 - 1}$ may be written as: $7 \left(\frac{1}{s^2 - 1^2} \right)$

Then $7 \left(\frac{1}{s^2 - 1^2} \right) = F(s) = \mathcal{L}\{7 \sinh t\}$ from (x) of Table 67.1, page 728

$$\begin{aligned} \text{Hence, } \mathcal{L}^{-1} \left\{ \frac{7e^{-3s}}{s^2 - 1} \right\} &= H(t - 3) \cdot 7 \sinh(t - 3) \\ &= 7H(t - 3) \cdot \sinh(t - 3) \end{aligned}$$

Now try the following Practice Exercise

Practise Exercise 256 Inverse Laplace transformer of Heaviside functions (Answers on page 870)

1. Determine $\mathcal{L}^{-1} \left\{ \frac{e^{-9s}}{s} \right\}$
2. Determine $\mathcal{L}^{-1} \left\{ \frac{4e^{-3s}}{s} \right\}$
3. Determine $\mathcal{L}^{-1} \left\{ \frac{2e^{-2s}}{s^2} \right\}$
4. Determine $\mathcal{L}^{-1} \left\{ \frac{5e^{-2s}}{s^2 + 1} \right\}$
5. Determine $\mathcal{L}^{-1} \left\{ \frac{3se^{-4s}}{s^2 + 16} \right\}$
6. Determine $\mathcal{L}^{-1} \left\{ \frac{6e^{-2s}}{s^2 - 1} \right\}$
7. Determine $\mathcal{L}^{-1} \left\{ \frac{3e^{-6s}}{s^3} \right\}$
8. Determine $\mathcal{L}^{-1} \left\{ \frac{2se^{-4s}}{s^2 - 16} \right\}$
9. Determine $\mathcal{L}^{-1} \left\{ \frac{2se^{-\frac{1}{2}s}}{s^2 + 5} \right\}$
10. Determine $\mathcal{L}^{-1} \left\{ \frac{4e^{-7s}}{s - 1} \right\}$

For fully worked solutions to each of the problems in Practice Exercises 254 to 256 in this chapter, go to the website:

www.routledge.com/cw/bird

Chapter 71

The solution of differential equations using Laplace transforms

Why it is important to understand: The solution of differential equations using Laplace transforms

Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods. This simplification in the solving of equations, coupled with the ability to directly implement electrical components in their transformed form, makes the use of Laplace transforms widespread in both electrical engineering and control systems engineering. The procedures explained in previous chapters are used in this chapter which demonstrates how differential equations are solved using Laplace transforms.

At the end of this chapter, you should be able to:

- understand the procedure to solve differential equations using Laplace transforms
- solve differential equations using Laplace transforms

71.1 Introduction

An alternative method of solving differential equations to that used in Chapters 49 to 54 is possible by using Laplace transforms.

71.2 Procedure to solve differential equations by using Laplace transforms

- (i) Take the Laplace transform of both sides of the differential equation by applying the formulae

for the Laplace transforms of derivatives (i.e. equations (3) and (4) of Chapter 68) and, where necessary, using a list of standard Laplace transforms, such as Tables 67.1 and 68.1 on pages 728 and 732.

- (ii) Put in the given initial conditions, i.e. $y(0)$ and $y'(0)$.
- (iii) Rearrange the equation to make $\mathcal{L}\{y\}$ the subject.
- (iv) Determine y by using, where necessary, partial fractions, and taking the inverse of each term by using Table 69.1 on page 738.

71.3 Worked problems on solving differential equations using Laplace transforms

Problem 1. Use Laplace transforms to solve the differential equation

$$2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 0, \text{ given that when } x=0, \\ y=4 \text{ and } \frac{dy}{dx}=9$$

This is the same problem as Problem 1 of Chapter 53, page 554 and a comparison of methods can be made. Using the above procedure:

$$(i) \quad 2\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + 5\mathcal{L}\left\{\frac{dy}{dx}\right\} - 3\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$2[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 5[s\mathcal{L}\{y\} - y(0)] - 3\mathcal{L}\{y\} = 0,$$

from equations (3) and (4) of Chapter 68.

$$(ii) \quad y(0)=4 \text{ and } y'(0)=9$$

$$\text{Thus } 2[s^2\mathcal{L}\{y\} - 4s - 9] + 5[s\mathcal{L}\{y\} - 4] - 3\mathcal{L}\{y\} = 0$$

$$\text{i.e. } 2s^2\mathcal{L}\{y\} - 8s - 18 + 5s\mathcal{L}\{y\} - 20 - 3\mathcal{L}\{y\} = 0$$

(iii) Rearranging gives:

$$(2s^2 + 5s - 3)\mathcal{L}\{y\} = 8s + 38$$

$$\text{i.e. } \mathcal{L}\{y\} = \frac{8s + 38}{2s^2 + 5s - 3}$$

$$(iv) \quad y = \mathcal{L}^{-1}\left\{\frac{8s + 38}{2s^2 + 5s - 3}\right\}$$

$$\frac{8s + 38}{2s^2 + 5s - 3} \equiv \frac{8s + 38}{(2s - 1)(s + 3)}$$

$$\equiv \frac{A}{2s - 1} + \frac{B}{s + 3} \\ \equiv \frac{A(s + 3) + B(2s - 1)}{(2s - 1)(s + 3)}$$

$$\text{Hence } 8s + 38 = A(s + 3) + B(2s - 1)$$

$$\text{When } s = \frac{1}{2}, 42 = 3\frac{1}{2}A, \text{ from which, } A = 12$$

$$\text{When } s = -3, 14 = -7B, \text{ from which, } B = -2$$

$$\text{Hence } y = \mathcal{L}^{-1}\left\{\frac{8s + 38}{2s^2 + 5s - 3}\right\} \\ = \mathcal{L}^{-1}\left\{\frac{12}{2s - 1} - \frac{2}{s + 3}\right\} \\ = \mathcal{L}^{-1}\left\{\frac{12}{2(s - \frac{1}{2})}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s + 3}\right\}$$

$$\text{Hence } y = 6e^{\frac{1}{2}x} - 2e^{-3x}, \text{ from (iii) of}$$

Table 69.1.

Problem 2. Use Laplace transforms to solve the differential equation:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0, \text{ given that when } x=0, y=3 \\ \text{and } \frac{dy}{dx}=7$$

This is the same as Problem 3 of Chapter 53, page 555. Using the above procedure:

$$(i) \quad \mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} + 6\mathcal{L}\left\{\frac{dy}{dx}\right\} + 13\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\text{Hence } [s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 6[s\mathcal{L}\{y\} - y(0)] + 13\mathcal{L}\{y\} = 0,$$

from equations (3) and (4) of Chapter 68.

$$(ii) \quad y(0)=3 \text{ and } y'(0)=7$$

$$\text{Thus } s^2\mathcal{L}\{y\} - 3s - 7 + 6s\mathcal{L}\{y\} - 18 + 13\mathcal{L}\{y\} = 0$$

(iii) Rearranging gives:

$$(s^2 + 6s + 13)\mathcal{L}\{y\} = 3s + 25$$

$$\text{i.e. } \mathcal{L}\{y\} = \frac{3s + 25}{s^2 + 6s + 13}$$

$$(iv) \quad y = \mathcal{L}^{-1} \left\{ \frac{3s + 25}{s^2 + 6s + 13} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3s + 25}{(s + 3)^2 + 2^2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3(s + 3) + 16}{(s + 3)^2 + 2^2} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{3(s + 3)}{(s + 3)^2 + 2^2} \right\}$$

$$+ \mathcal{L}^{-1} \left\{ \frac{8(2)}{(s + 3)^2 + 2^2} \right\}$$

$$= 3e^{-3t} \cos 2t + 8e^{-3t} \sin 2t, \text{ from (xiii)}$$

and (xii) of Table 69.1

$$\text{Hence } y = e^{-3t}(3 \cos 2t + 8 \sin 2t)$$

Problem 3. Use Laplace transforms to solve the differential equation:

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} = 9, \text{ given that when } x = 0, y = 0 \text{ and } \frac{dy}{dx} = 0$$

This is the same problem as Problem 2 of Chapter 54, page 562. Using the procedure:

$$(i) \quad \mathcal{L} \left\{ \frac{d^2 y}{dx^2} \right\} - 3\mathcal{L} \left\{ \frac{dy}{dx} \right\} = \mathcal{L}\{9\}$$

$$\text{Hence } [s^2 \mathcal{L}\{y\} - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\} - y(0)] = \frac{9}{s}$$

$$(ii) \quad y(0) = 0 \text{ and } y'(0) = 0$$

$$\text{Hence } s^2 \mathcal{L}\{y\} - 3s\mathcal{L}\{y\} = \frac{9}{s}$$

(iii) Rearranging gives:

$$(s^2 - 3s)\mathcal{L}\{y\} = \frac{9}{s}$$

$$\text{i.e. } \mathcal{L}\{y\} = \frac{9}{s(s^2 - 3s)} = \frac{9}{s^2(s - 3)}$$

$$(iv) \quad y = \mathcal{L}^{-1} \left\{ \frac{9}{s^2(s - 3)} \right\}$$

$$\frac{9}{s^2(s - 3)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 3}$$

$$\equiv \frac{A(s)(s - 3) + B(s - 3) + Cs^2}{s^2(s - 3)}$$

$$\text{Hence } 9 \equiv A(s)(s - 3) + B(s - 3) + Cs^2$$

$$\text{When } s = 0, 9 = -3B, \text{ from which, } B = -3$$

$$\text{When } s = 3, 9 = 9C, \text{ from which, } C = 1$$

$$\text{Equating } s^2 \text{ terms gives: } 0 = A + C, \text{ from which, } A = -1, \text{ since } C = 1. \text{ Hence,}$$

$$\mathcal{L}^{-1} \left\{ \frac{9}{s^2(s - 3)} \right\} = \mathcal{L}^{-1} \left\{ -\frac{1}{s} - \frac{3}{s^2} + \frac{1}{s - 3} \right\}$$

$$= -1 - 3x + e^{3x}, \text{ from (i),}$$

(vi) and (iii) of Table 69.1.

$$\text{i.e. } y = e^{3x} - 3x - 1$$

Problem 4. Use Laplace transforms to solve the differential equation:

$$\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 10y = e^{2x} + 20, \text{ given that when } x = 0, y = 0 \text{ and } \frac{dy}{dx} = -\frac{1}{3}$$

Using the procedure:

$$(i) \quad \mathcal{L} \left\{ \frac{d^2 y}{dx^2} \right\} - 7\mathcal{L} \left\{ \frac{dy}{dx} \right\} + 10\mathcal{L}\{y\} = \mathcal{L}\{e^{2x} + 20\}$$

$$\text{Hence } [s^2 \mathcal{L}\{y\} - sy(0) - y'(0)] - 7[s\mathcal{L}\{y\} - y(0)] + 10\mathcal{L}\{y\} = \frac{1}{s - 2} + \frac{20}{s}$$

$$(ii) \quad y(0) = 0 \text{ and } y'(0) = -\frac{1}{3}$$

$$\text{Hence } s^2 \mathcal{L}\{y\} - 0 - \left(-\frac{1}{3}\right) - 7s\mathcal{L}\{y\} + 0 + 10\mathcal{L}\{y\} = \frac{21s - 40}{s(s - 2)}$$

$$(iii) \quad (s^2 - 7s + 10)\mathcal{L}\{y\} = \frac{21s - 40}{s(s - 2)} - \frac{1}{3} = \frac{3(21s - 40) - s(s - 2)}{3s(s - 2)}$$

$$= \frac{-s^2 + 65s - 120}{3s(s-2)}$$

Hence $\mathcal{L}\{y\} = \frac{-s^2 + 65s - 120}{3s(s-2)(s^2 - 7s + 10)}$

$$= \frac{1}{3} \left[\frac{-s^2 + 65s - 120}{s(s-2)(s-2)(s-5)} \right]$$

$$= \frac{1}{3} \left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2} \right]$$

$$(iv) \quad y = \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2} \right\}$$

$$\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}$$

$$\equiv \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$\equiv \frac{\left(\begin{array}{l} A(s-5)(s-2)^2 + B(s)(s-2)^2 \\ + C(s)(s-5)(s-2) + D(s)(s-5) \end{array} \right)}{s(s-5)(s-2)^2}$$

Hence

$$-s^2 + 65s - 120$$

$$\equiv A(s-5)(s-2)^2 + B(s)(s-2)^2$$

$$+ C(s)(s-5)(s-2) + D(s)(s-5)$$

When $s=0$, $-120 = -20A$, from which, $A=6$

When $s=5$, $180 = 45B$, from which, $B=4$

When $s=2$, $6 = -6D$, from which, $D=-1$

Equating s^3 terms gives: $0 = A + B + C$, from which, $C = -10$

Hence $\frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2} \right\}$

$$= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{6}{s} + \frac{4}{s-5} - \frac{10}{s-2} - \frac{1}{(s-2)^2} \right\}$$

$$= \frac{1}{3} [6 + 4e^{5x} - 10e^{2x} - xe^{2x}]$$

Thus $y = 2 + \frac{4}{3}e^{5x} - \frac{10}{3}e^{2x} - \frac{x}{3}e^{2x}$

Problem 5. The current flowing in an electrical circuit is given by the differential equation

$Ri + L(di/dt) = E$, where E , L and R are constants. Use Laplace transforms to solve the equation for current i given that when $t=0$, $i=0$

Using the procedure:

$$(i) \quad \mathcal{L}\{Ri\} + \mathcal{L}\left\{L \frac{di}{dt}\right\} = \mathcal{L}\{E\}$$

$$\text{i.e. } R\mathcal{L}\{i\} + L[s\mathcal{L}\{i\} - i(0)] = \frac{E}{s}$$

$$(ii) \quad i(0)=0, \text{ hence } R\mathcal{L}\{i\} + Ls\mathcal{L}\{i\} = \frac{E}{s}$$

(iii) Rearranging gives:

$$(R + Ls)\mathcal{L}\{i\} = \frac{E}{s}$$

$$\text{i.e. } \mathcal{L}\{i\} = \frac{E}{s(R + Ls)}$$

$$(iv) \quad i = \mathcal{L}^{-1} \left\{ \frac{E}{s(R + Ls)} \right\}$$

$$\frac{E}{s(R + Ls)} \equiv \frac{A}{s} + \frac{B}{R + Ls}$$

$$\equiv \frac{A(R + Ls) + Bs}{s(R + Ls)}$$

Hence $E = A(R + Ls) + Bs$

When $s=0$, $E = AR$,

from which, $A = \frac{E}{R}$

When $s = -\frac{R}{L}$, $E = B\left(-\frac{R}{L}\right)$

from which, $B = -\frac{EL}{R}$

Hence $\mathcal{L}^{-1} \left\{ \frac{E}{s(R + Ls)} \right\}$

$$= \mathcal{L}^{-1} \left\{ \frac{E/R}{s} + \frac{-EL/R}{R + Ls} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{E}{Rs} - \frac{EL}{R(R + Ls)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{E}{R} \left(\frac{1}{s} \right) - \frac{E}{R} \left(\frac{1}{\frac{R}{L} + s} \right) \right\}$$

$$= \frac{E}{R} \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{\left(s + \frac{R}{L}\right)} \right\}$$

Hence current $i = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$

Now try the following Practice Exercise

Practice Exercise 257 Solving differential equations using Laplace transforms
(Answers on page 870)

1. A first-order differential equation involving current i in a series R – L circuit is given by:
 $\frac{di}{dt} + 5i = \frac{E}{2}$ and $i = 0$ at time $t = 0$
Use Laplace transforms to solve for i when (a) $E = 20$ (b) $E = 40e^{-3t}$ and (c) $E = 50 \sin 5t$

In Problems 2 to 9, use Laplace transforms to solve the given differential equations.

2. $9 \frac{d^2 y}{dt^2} - 24 \frac{dy}{dt} + 16y = 0$, given $y(0) = 3$ and $y'(0) = 3$
3. $\frac{d^2 x}{dt^2} + 100x = 0$, given $x(0) = 2$ and $x'(0) = 0$
4. $\frac{d^2 i}{dt^2} + 1000 \frac{di}{dt} + 250\,000i = 0$, given $i(0) = 0$ and $i'(0) = 100$
5. $\frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 8x = 0$, given $x(0) = 4$ and $x'(0) = 8$

6. $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 3e^{4x}$, given $y(0) = -\frac{2}{3}$ and $y'(0) = 4\frac{1}{3}$

7. $\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} - 4y = 3 \sin x$, given $y(0) = 0$ and $y'(0) = 0$

8. $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = 3 \cos 3x - 11 \sin 3x$, given $y(0) = 0$ and $y'(0) = 6$

9. $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 3e^x \cos 2x$, given $y(0) = 2$ and $y'(0) = 5$

10. The free oscillations of a lightly damped elastic system are given by the equation:

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 5y = 0$$

where y is the displacement from the equilibrium position. If when time $t = 0$, $y = 2$ and $\frac{dy}{dt} = 0$, determine an expression for the displacement.

11. Solve, using Laplace transforms, Problems 4 to 9 of Exercise 203, page 556 and Problems 1 to 5 of Exercise 204, page 558.
12. Solve, using Laplace transforms, Problems 3 to 6 of Exercise 205, page 563, Problems 5 and 6 of Exercise 206, page 565, Problems 4 and 7 of Exercise 207, page 567 and Problems 5 and 6 of Exercise 208, page 569.

For fully worked solutions to each of the problems in Practice Exercise 257 in this chapter, go to the website:

www.routledge.com/cw/bird

The solution of simultaneous differential equations using Laplace transforms

Why it is important to understand: The solution of simultaneous differential equations using Laplace transforms

As stated in previous chapters, Laplace transforms have many applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability and Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods. Specifically, this chapter explains the procedure for solving simultaneous differential equations; this requires all of the knowledge gained in the preceding chapters.

At the end of this chapter, you should be able to:

- understand the procedure to solve simultaneous differential equations using Laplace transforms
- solve simultaneous differential equations using Laplace transforms

72.1 Introduction

It is sometimes necessary to solve simultaneous differential equations. An example occurs when two electrical circuits are coupled magnetically where the equations relating the two currents i_1 and i_2 are typically:

$$\begin{aligned}L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + R_1 i_1 &= E_1 \\L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + R_2 i_2 &= 0\end{aligned}$$

where L represents inductance, R resistance, M mutual inductance and E_1 the p.d. applied to one of the circuits.

72.2 Procedure to solve simultaneous differential equations using Laplace transforms

- (i) Take the Laplace transform of both sides of each simultaneous equation by applying the formulae for the Laplace transforms of derivatives (i.e. equations (3) and (4) of Chapter 68, page 734) and

using a list of standard Laplace transforms, as in Table 67.1, page 728 and Table 68.1, page 732.

- (ii) Put in the initial conditions, i.e. $x(0)$, $y(0)$, $x'(0)$, $y'(0)$
- (iii) Solve the simultaneous equations for $\mathcal{L}\{y\}$ and $\mathcal{L}\{x\}$ by the normal algebraic method.
- (iv) Determine y and x by using, where necessary, partial fractions, and taking the inverse of each term.

72.3 Worked problems on solving simultaneous differential equations by using Laplace transforms

Problem 1. Solve the following pair of simultaneous differential equations

$$\frac{dy}{dt} + x = 1$$

$$\frac{dx}{dt} - y + 4e^t = 0$$

given that at $t=0$, $x=0$ and $y=0$

Using the above procedure:

$$(i) \quad \mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{1\} \quad (1)$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{y\} + 4\mathcal{L}\{e^t\} = 0 \quad (2)$$

Equation (1) becomes:

$$[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{x\} = \frac{1}{s} \quad (1')$$

from equation (3), page 734 and Table 67.1, page 728.

Equation (2) becomes:

$$[s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{4}{s-1} \quad (2')$$

- (ii) $x(0)=0$ and $y(0)=0$ hence

Equation (1') becomes:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \quad (1'')$$

and equation (2') becomes:

$$s\mathcal{L}\{x\} - \mathcal{L}\{y\} = -\frac{4}{s-1}$$

$$\text{or } -\mathcal{L}\{y\} + s\mathcal{L}\{x\} = -\frac{4}{s-1} \quad (2'')$$

- (iii) $1 \times$ equation (1'') and $s \times$ equation (2'') gives:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \quad (3)$$

$$-s\mathcal{L}\{y\} + s^2\mathcal{L}\{x\} = -\frac{4s}{s-1} \quad (4)$$

Adding equations (3) and (4) gives:

$$\begin{aligned} (s^2 + 1)\mathcal{L}\{x\} &= \frac{1}{s} - \frac{4s}{s-1} \\ &= \frac{(s-1) - s(4s)}{s(s-1)} \\ &= \frac{-4s^2 + s - 1}{s(s-1)} \end{aligned}$$

$$\text{from which, } \mathcal{L}\{x\} = \frac{-4s^2 + s - 1}{s(s-1)(s^2 + 1)} \quad (5)$$

Using partial fractions

$$\begin{aligned} \frac{-4s^2 + s - 1}{s(s-1)(s^2 + 1)} &\equiv \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs + D}{(s^2 + 1)} \\ &= \frac{\left(A(s-1)(s^2 + 1) + Bs(s^2 + 1) + (Cs + D)s(s-1) \right)}{s(s-1)(s^2 + 1)} \end{aligned}$$

Hence

$$\begin{aligned} -4s^2 + s - 1 &= A(s-1)(s^2 + 1) + Bs(s^2 + 1) \\ &\quad + (Cs + D)s(s-1) \end{aligned}$$

$$\text{When } s=0, \quad -1 = -A \quad \text{hence } A=1$$

$$\text{When } s=1, \quad -4 = 2B \quad \text{hence } B=-2$$

Equating s^3 coefficients:

$$\begin{aligned} 0 &= A + B + C \quad \text{hence } C=1 \\ &\quad (\text{since } A=1 \text{ and } B=-2) \end{aligned}$$

Equating s^2 coefficients:

$$\begin{aligned} -4 &= -A + D - C \quad \text{hence } D=-2 \\ &\quad (\text{since } A=1 \text{ and } C=1) \end{aligned}$$

$$\begin{aligned}\text{Thus } \mathcal{L}\{x\} &= \frac{-4s^2 + s - 1}{s(s-1)(s^2+1)} \\ &= \frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)}\end{aligned}$$

(iv) Hence

$$\begin{aligned}x &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{2}{(s-1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)} \right\}\end{aligned}$$

$$\text{i.e. } x = 1 - 2e^t + \cos t - 2 \sin t,$$

from Table 69.1, page 738

From the second equation given in the question,

$$\frac{dx}{dt} - y + 4e^t = 0$$

from which,

$$\begin{aligned}y &= \frac{dx}{dt} + 4e^t \\ &= \frac{d}{dt}(1 - 2e^t + \cos t - 2 \sin t) + 4e^t \\ &= -2e^t - \sin t - 2 \cos t + 4e^t\end{aligned}$$

$$\text{i.e. } y = 2e^t - \sin t - 2 \cos t$$

[Alternatively, to determine y , return to equations (1'') and (2'')]**Problem 2.** Solve the following pair of simultaneous differential equations

$$3 \frac{dx}{dt} - 5 \frac{dy}{dt} + 2x = 6$$

$$2 \frac{dy}{dt} - \frac{dx}{dt} - y = -1$$

given that at $t=0$, $x=8$ and $y=3$

Using the above procedure:

$$(i) \quad 3\mathcal{L}\left\{\frac{dx}{dt}\right\} - 5\mathcal{L}\left\{\frac{dy}{dt}\right\} + 2\mathcal{L}\{x\} = \mathcal{L}\{6\} \quad (1)$$

$$2\mathcal{L}\left\{\frac{dy}{dt}\right\} - \mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{y\} = \mathcal{L}\{-1\} \quad (2)$$

Equation (1) becomes:

$$3[s\mathcal{L}\{x\} - x(0)] - 5[s\mathcal{L}\{y\} - y(0)] + 2\mathcal{L}\{x\} = \frac{6}{s}$$

from equation (3), page 734, and Table 67.1, page 728.

$$\begin{aligned}\text{i.e. } 3s\mathcal{L}\{x\} - 3x(0) - 5s\mathcal{L}\{y\} \\ + 5y(0) + 2\mathcal{L}\{x\} &= \frac{6}{s} \\ \text{i.e. } (3s+2)\mathcal{L}\{x\} - 3x(0) - 5s\mathcal{L}\{y\} \\ + 5y(0) &= \frac{6}{s} \quad (1')\end{aligned}$$

Equation (2) becomes:

$$2[s\mathcal{L}\{y\} - y(0)] - [s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{1}{s}$$

from equation (3), page 734, and Table 67.1, page 728,

$$\begin{aligned}\text{i.e. } 2s\mathcal{L}\{y\} - 2y(0) - s\mathcal{L}\{x\} \\ + x(0) - \mathcal{L}\{y\} &= -\frac{1}{s} \\ \text{i.e. } (2s-1)\mathcal{L}\{y\} - 2y(0) - s\mathcal{L}\{x\} \\ + x(0) &= -\frac{1}{s} \quad (2')\end{aligned}$$

(ii) $x(0)=8$ and $y(0)=3$, hence equation (1') becomes

$$\begin{aligned}(3s+2)\mathcal{L}\{x\} - 3(8) - 5s\mathcal{L}\{y\} \\ + 5(3) &= \frac{6}{s} \quad (1'')\end{aligned}$$

and equation (2') becomes

$$\begin{aligned}(2s-1)\mathcal{L}\{y\} - 2(3) - s\mathcal{L}\{x\} \\ + 8 &= -\frac{1}{s} \quad (2'')\end{aligned}$$

$$\text{i.e. } (3s+2)\mathcal{L}\{x\} - 5s\mathcal{L}\{y\} = \frac{6}{s} + 9 \quad (1''')$$

$$\left. \begin{aligned} (3s+2)\mathcal{L}\{x\} - 5s\mathcal{L}\{y\} &= \frac{6}{s} + 9 \quad (1''') \\ -s\mathcal{L}\{x\} + (2s-1)\mathcal{L}\{y\} &= -\frac{1}{s} - 2 \quad (2''') \end{aligned} \right\} \quad (A)$$

(iii) $s \times$ equation (1''') and $(3s + 2) \times$ equation (2''') gives:

$$s(3s + 2)\mathcal{L}\{x\} - 5s^2\mathcal{L}\{y\} = s\left(\frac{6}{s} + 9\right) \quad (3)$$

$$\begin{aligned} -s(3s + 2)\mathcal{L}\{x\} + (3s + 2)(2s - 1)\mathcal{L}\{y\} \\ = (3s + 2)\left(-\frac{1}{s} - 2\right) \end{aligned} \quad (4)$$

$$\text{i.e. } s(3s + 2)\mathcal{L}\{x\} - 5s^2\mathcal{L}\{y\} = 6 + 9s \quad (3')$$

$$\begin{aligned} -s(3s + 2)\mathcal{L}\{x\} + (6s^2 + s - 2)\mathcal{L}\{y\} \\ = -6s - \frac{2}{s} - 7 \end{aligned} \quad (4')$$

Adding equations (3') and (4') gives:

$$\begin{aligned} (s^2 + s - 2)\mathcal{L}\{y\} &= -1 + 3s - \frac{2}{s} \\ &= \frac{-s + 3s^2 - 2}{s} \end{aligned}$$

$$\text{from which, } \mathcal{L}\{y\} = \frac{3s^2 - s - 2}{s(s^2 + s - 2)}$$

Using partial fractions

$$\begin{aligned} \frac{3s^2 - s - 2}{s(s^2 + s - 2)} \\ \equiv \frac{A}{s} + \frac{B}{(s + 2)} + \frac{C}{(s - 1)} \\ = \frac{A(s + 2)(s - 1) + Bs(s - 1) + Cs(s + 2)}{s(s + 2)(s - 1)} \end{aligned}$$

$$\begin{aligned} \text{i.e. } 3s^2 - s - 2 &= A(s + 2)(s - 1) \\ &\quad + Bs(s - 1) + Cs(s + 2) \end{aligned}$$

$$\text{When } s = 0, -2 = -2A, \text{ hence } A = 1$$

$$\text{When } s = 1, 0 = 3C, \text{ hence } C = 0$$

$$\text{When } s = -2, 12 = 6B, \text{ hence } B = 2$$

$$\text{Thus } \mathcal{L}\{y\} = \frac{3s^2 - s - 2}{s(s^2 + s - 2)} = \frac{1}{s} + \frac{2}{(s + 2)}$$

$$(iv) \text{ Hence } y = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s + 2}\right\} = 1 + 2e^{-2t}$$

Returning to equations (A) to determine $\mathcal{L}\{x\}$ and hence x :

$(2s - 1) \times$ equation (1''') and $5s \times$ (2''') gives:

$$\begin{aligned} (2s - 1)(3s + 2)\mathcal{L}\{x\} - 5s(2s - 1)\mathcal{L}\{y\} \\ = (2s - 1)\left(\frac{6}{s} + 9\right) \end{aligned} \quad (5)$$

$$\begin{aligned} \text{and } -s(5s)\mathcal{L}\{x\} + 5s(2s - 1)\mathcal{L}\{y\} \\ = 5s\left(-\frac{1}{s} - 2\right) \end{aligned} \quad (6)$$

$$\begin{aligned} \text{i.e. } (6s^2 + s - 2)\mathcal{L}\{x\} - 5s(2s - 1)\mathcal{L}\{y\} \\ = 12 + 18s - \frac{6}{s} - 9 \end{aligned} \quad (5')$$

$$\begin{aligned} \text{and } -5s^2\mathcal{L}\{x\} + 5s(2s - 1)\mathcal{L}\{y\} \\ = -5 - 10s \end{aligned} \quad (6')$$

Adding equations (5') and (6') gives:

$$\begin{aligned} (s^2 + s - 2)\mathcal{L}\{x\} &= -2 + 8s - \frac{6}{s} \\ &= \frac{-2s + 8s^2 - 6}{s} \end{aligned}$$

$$\begin{aligned} \text{from which, } \mathcal{L}\{x\} &= \frac{8s^2 - 2s - 6}{s(s^2 + s - 2)} \\ &= \frac{8s^2 - 2s - 6}{s(s + 2)(s - 1)} \end{aligned}$$

Using partial fractions

$$\begin{aligned} \frac{8s^2 - 2s - 6}{s(s + 2)(s - 1)} \\ \equiv \frac{A}{s} + \frac{B}{(s + 2)} + \frac{C}{(s - 1)} \\ = \frac{A(s + 2)(s - 1) + Bs(s - 1) + Cs(s + 2)}{s(s + 2)(s - 1)} \end{aligned}$$

$$\begin{aligned} \text{i.e. } 8s^2 - 2s - 6 &= A(s + 2)(s - 1) \\ &\quad + Bs(s - 1) + Cs(s + 2) \end{aligned}$$

When $s=0$, $-6=-2A$, hence $A=3$

When $s=1$, $0=3C$, hence $C=0$

When $s=-2$, $30=6B$, hence $B=5$

$$\text{Thus } \mathcal{L}\{x\} = \frac{8s^2 - 2s - 6}{s(s+2)(s-1)} = \frac{3}{s} + \frac{5}{(s+2)}$$

$$\text{Hence } x = \mathcal{L}^{-1} \left\{ \frac{3}{s} + \frac{5}{s+2} \right\} = 3 + 5e^{-2t}$$

Therefore the solutions of the given simultaneous differential equations are

$$y = 1 + 2e^{-2t} \quad \text{and} \quad x = 3 + 5e^{-2t}$$

(These solutions may be checked by substituting the expressions for x and y into the original equations.)

Problem 3. Solve the following pair of simultaneous differential equations

$$\frac{d^2x}{dt^2} - x = y$$

$$\frac{d^2y}{dt^2} + y = -x$$

given that at $t=0$, $x=2$, $y=-1$, $\frac{dx}{dt}=0$

and $\frac{dy}{dt}=0$

Using the procedure:

$$(i) \quad [s^2 \mathcal{L}\{x\} - sx(0) - x'(0)] - \mathcal{L}\{x\} = \mathcal{L}\{y\} \quad (1)$$

$$[s^2 \mathcal{L}\{y\} - sy(0) - y'(0)] + \mathcal{L}\{y\} = -\mathcal{L}\{x\} \quad (2)$$

from equation (4), page 734

$$(ii) \quad x(0)=2, y(0)=-1, x'(0)=0 \text{ and } y'(0)=0$$

$$\text{hence } s^2 \mathcal{L}\{x\} - 2s - \mathcal{L}\{x\} = \mathcal{L}\{y\} \quad (1')$$

$$s^2 \mathcal{L}\{y\} + s + \mathcal{L}\{y\} = -\mathcal{L}\{x\} \quad (2')$$

(iii) Rearranging gives:

$$(s^2 - 1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s \quad (3)$$

$$\mathcal{L}\{x\} + (s^2 + 1)\mathcal{L}\{y\} = -s \quad (4)$$

Equation (3) $\times (s^2 + 1)$ and equation (4) $\times 1$ gives:

$$(s^2 + 1)(s^2 - 1)\mathcal{L}\{x\} - (s^2 + 1)\mathcal{L}\{y\} = (s^2 + 1)2s \quad (5)$$

$$\mathcal{L}\{x\} + (s^2 + 1)\mathcal{L}\{y\} = -s \quad (6)$$

Adding equations (5) and (6) gives:

$$[(s^2 + 1)(s^2 - 1) + 1]\mathcal{L}\{x\} = (s^2 + 1)2s - s$$

$$\text{i.e. } s^4 \mathcal{L}\{x\} = 2s^3 + s = s(2s^2 + 1)$$

$$\text{from which, } \mathcal{L}\{x\} = \frac{s(2s^2 + 1)}{s^4} = \frac{2s^2 + 1}{s^3}$$

$$= \frac{2s^2}{s^3} + \frac{1}{s^3} = \frac{2}{s} + \frac{1}{s^3}$$

$$(iv) \quad \text{Hence } x = \mathcal{L}^{-1} \left\{ \frac{2}{s} + \frac{1}{s^3} \right\}$$

$$\text{i.e. } x = 2 + \frac{1}{2}t^2$$

Returning to equations (3) and (4) to determine y :

$1 \times$ equation (3) and $(s^2 - 1) \times$ equation (4) gives:

$$(s^2 - 1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s \quad (7)$$

$$(s^2 - 1)\mathcal{L}\{x\} + (s^2 - 1)(s^2 + 1)\mathcal{L}\{y\} = -s(s^2 - 1) \quad (8)$$

Equation (7) $-$ equation (8) gives:

$$[-1 - (s^2 - 1)(s^2 + 1)]\mathcal{L}\{y\} = 2s + s(s^2 - 1)$$

$$\text{i.e. } -s^4 \mathcal{L}\{y\} = s^3 + s$$

$$\text{and } \mathcal{L}\{y\} = \frac{s^3 + s}{-s^4} = -\frac{1}{s} - \frac{1}{s^3}$$

$$\text{from which, } y = \mathcal{L}^{-1} \left\{ -\frac{1}{s} - \frac{1}{s^3} \right\}$$

$$\text{i.e. } y = -1 - \frac{1}{2}t^2$$

Now try the following Practice Exercise

Practice Exercise 258 Solving simultaneous differential equations using Laplace transforms (Answers on page 870)

Solve the following pairs of simultaneous differential equations:

1. $2\frac{dx}{dt} + \frac{dy}{dt} = 5e^t$

$$\frac{dy}{dt} - 3\frac{dx}{dt} = 5$$

given that when $t=0$, $x=0$ and $y=0$

2. $2\frac{dy}{dt} - y + x + \frac{dx}{dt} - 5\sin t = 0$

$$3\frac{dy}{dt} + x - y + 2\frac{dx}{dt} - e^t = 0$$

given that at $t=0$, $x=0$ and $y=0$

3. $\frac{d^2x}{dt^2} + 2x = y$

$$\frac{d^2y}{dt^2} + 2y = x$$

given that at $t=0$, $x=4$, $y=2$, $\frac{dx}{dt}=0$

and $\frac{dy}{dt}=0$

For fully worked solutions to each of the problems in Practice Exercise 258 in this chapter,
go to the website:

www.routledge.com/cw/bird



Revision Test 19 Laplace transforms

This Revision Test covers the material contained in Chapters 67 to 72. *The marks for each question are shown in brackets at the end of each question.*

1. Find the Laplace transforms of the following functions:
 - (a) $2t^3 - 4t + 5$ (b) $3e^{-2t} - 4\sin 2t$
 - (c) $3\cosh 2t$ (d) $2t^4 e^{-3t}$
 - (e) $5e^{2t} \cos 3t$ (f) $2e^{3t} \sinh 4t$ (16)
2. Find the inverse Laplace transforms of the following functions:
 - (a) $\frac{5}{2s+1}$ (b) $\frac{12}{s^5}$
 - (c) $\frac{4s}{s^2+9}$ (d) $\frac{5}{s^2-9}$
 - (e) $\frac{3}{(s+2)^4}$ (f) $\frac{s-4}{s^2-8s-20}$
 - (g) $\frac{8}{s^2-4s+3}$ (17)
3. Use partial fractions to determine the following:
 - (a) $\mathcal{L}^{-1} \left\{ \frac{5s-1}{s^2-s-2} \right\}$
 - (b) $\mathcal{L}^{-1} \left\{ \frac{2s^2+11s-9}{s(s-1)(s+3)} \right\}$
 - (c) $\mathcal{L}^{-1} \left\{ \frac{13-s^2}{s(s^2+4s+13)} \right\}$ (24)
4. Determine the poles and zeros for the transfer function: $F(s) = \frac{(s+2)(s-3)}{(s+3)(s^2+2s+5)}$ and plot them on a pole-zero diagram. (10)
5. Sketch the graphs of (a) $f(t) = 4H(t-3)$
 (b) $f(t) = 3[H(t-2) - H(t-5)]$ (5)
6. Determine (a) $\mathcal{L}\{H(t-2).e^{t-2}\}$
 (b) $\mathcal{L}\{5H(t-1).\sin(t-1)\}$ (6)
7. Determine (a) $\mathcal{L}^{-1} \left\{ \frac{3e^{-2s}}{s^2} \right\}$ (b) $\mathcal{L}^{-1} \left\{ \frac{3se^{-5s}}{s^2+4} \right\}$ (9)
8. In a galvanometer the deflection θ satisfies the differential equation:

$$\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt} + \theta = 4$$
 Use Laplace transforms to solve the equation for θ given that when $t=0$, $\theta=0$ and $\frac{d\theta}{dt}=0$ (13)
9. Solve the following pair of simultaneous differential equations:

$$3\frac{dx}{dt} = 3x + 2y$$

$$2\frac{dy}{dt} + 3x = 6y$$
 given that when $t=0$, $x=1$ and $y=3$ (20)

For lecturers/instructors/teachers, fully worked solutions to each of the problems in Revision Test 19, together with a full marking scheme, are available at the website:

www.routledge.com/cw/bird